Reaction-Advection-Diffusion Equations

Most models of biological/physical processes in the ocean build upon four fundamental processes

- advection: movement by the ocean currents or by the organisms
- diffusion: mixing processes, whether by molecular motion, turbulence, or movement of organisms/ material relative to the water
- reaction: processes which transfer energy/ biomass/ carbon/... from one group of organisms (or other categories such as dissolved CO_2 or non-living particulates) to another. These are generally framed as though they were kinds of chemical reactions; the rates at which the concentration of one component changes is a function of the concentrations of this property and of its resources or prey and its consumers or predators.
- boundary forcing: winds, fluxes of heat or freshwater, and tides which generate flows and alter the temperature and salinity structure. Light, fluxes of carbon, oxygen, dust/iron, etc. all affect the biology.

The first three of these lead to the basic PDE's governing the interior dynamics. These are the reaction-advection-diffusion equations

$$\frac{\partial}{\partial t}b_i + \nabla \cdot (\mathbf{u}_i b_i) = \nabla \cdot \kappa_i \nabla b_i + \mathcal{B}_i(b_1, b_2, b_3, \ldots, \mathbf{x}, t) \tag{1}$$

where $b_i(\mathbf{x}, t)$ is the density of the ith class of organisms, $\mathcal{B}_i$ represents the processes which transfer biomass or carbon/ nitrogen mass to this class from the others, κ is the diffusivity, and $\mathbf{u}$ is the advecting velocity.[†] We'll abbreviate the reaction terms as $\mathcal{B}(b, \mathbf{x}, t)$.

trajectories
diffusion

Outline

- basics of advection
    - local flow properties
- basics of diffusion
- stability
- conservation and homogenization
- bio dominance
- mixing
    - bio fronts
    - spatially variable growth/ instability
- advection
    - washout

[†] We shall use boldface such as $\mathbf{x}$ to indicate vectors in ordinary three-dimensional space, with indices or special variable names used as convenient. Thus $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$ gives the east, north, and upward distances. Velocities $\mathbf{u} = (u, v, w)$ are denoted similarly.

Basics of advection

We can rephrase the basic equation in terms of the $\frac{D}{Dt}$ operator

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_i \cdot \nabla\right) b_i - \nabla \cdot \kappa_i \nabla b_i = -b_i \nabla \cdot \mathbf{u}_i + \mathcal{B}_i(b_1, b_2, b_3, \ldots, \mathbf{x}, t)$$

Note that $\mathbf{u}_i$ includes any movement relative to the water (with the random part included in κ_i; for organisms which are nearly planktonic, it makes sense to separate out these movements and write $\mathbf{u}_i = \mathbf{u} + \mathbf{v}_i$. Applying $\nabla \cdot \mathbf{u} = 0$ gives

$$\frac{D}{Dt} b_i - \nabla \cdot \kappa_i \nabla b_i = -\nabla \cdot (\mathbf{v}_i b_i) + \mathcal{B}_i$$

Convergence (divergence) of the movement-induced fluxes thus acts as a source (sink) of local concentration.

Local flow properties

We now consider the ways in which advection work, harking back to the example above to think about how a local volume of fluid behaves. As we saw, it can move, stretch in some dimensions and shrink in others, and rotate. To see this more clearly, consider what happens to a line joining two points when the points move with the fluid. The first point is at $\mathbf{X}(t)$ and satisfies

$$\frac{\partial}{\partial t}\mathbf{X}(t) = \mathbf{u}(\mathbf{X}(t), t)$$

The second point at $\mathbf{X}(t) + d\mathbf{X}(t)$ evolves according to

$$\frac{\partial}{\partial t}\mathbf{X}(t) + \frac{\partial}{\partial t}d\mathbf{X}(t) = \mathbf{u}(\mathbf{X}(t) + d\mathbf{X}(t), t)$$

or

$$\frac{\partial}{\partial t}d\mathbf{X}(t) = \mathbf{u}(\mathbf{X}(t) + d\mathbf{X}(t), t) - \mathbf{u}(\mathbf{X}(t), t)$$

For small separations,

$$\frac{\partial}{\partial t}dX_i \simeq \sum_j \frac{\partial u_i}{\partial x_j}dX_j$$

Henceforth, the sum will be implicit: if a repeated index appears on the right, but not the left, it is summed over (Einstein notation).

example movement divergence strain rotation strain+rotation

There is, then, a tendency for the whole line segment to translate – both ends have a $\mathbf{u}(\mathbf{X}, t)$ component. In addition, the segment changes by the action of the rate of strain matrix

$$\frac{\partial}{\partial t}X_i = M_{ij}X_j \quad , \quad \mathbf{M} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$$

We can split off the trace of $\mathbf{M}$, which is just the divergence of the flow, and divide the remaining matrix into symmetric and an antisymmetric pieces

$$\mathbf{M} = \frac{\nabla \cdot \mathbf{u}}{3}\,\mathbf{I} + \mathbf{S} + \mathbf{R}$$

with

$$\mathbf{S} = \begin{pmatrix} \frac{2}{3}\frac{\partial u}{\partial x} - \frac{1}{3}\frac{\partial v}{\partial y} - \frac{1}{3}\frac{\partial w}{\partial z} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{2}{3}\frac{\partial v}{\partial y} - \frac{1}{3}\frac{\partial w}{\partial z} - \frac{1}{3}\frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} - \frac{1}{3}\frac{\partial u}{\partial x} - \frac{1}{3}\frac{\partial v}{\partial y} \end{pmatrix}$$

$$\mathbf{R} = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} & \frac{\partial u}{\partial w} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} & 0 & \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} & 0 \end{pmatrix}$$

The first term represents uniform expansion or contraction. The second is pure strain: as a real, symmetric matrix, it can be diagonalized in a suitable reference frame

$$\mathbf{S}' = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix}$$

with $s_1 + s_2 + s_3 = 0$. Thus we have contraction along one or two axes (the negative s values) and expansion along the others.

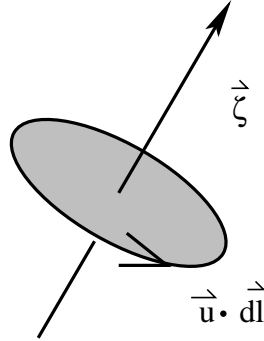The third matrix represents rotation:

$$R_{ij}\,dX_j = \frac{1}{2}(\boldsymbol{\zeta} \times d\mathbf{X})_i$$

with

$$\boldsymbol{\zeta} = \nabla \times \mathbf{u} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\,,\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\,,\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Vorticity

The curl of the velocity, called the vorticity, plays a central role in geophysical flows, so it is worthwhile examining it more closely.

Vorticity and circulation

If we take a small disk perpendicular to the vorticity vector, Stokes' theorem says that

$$\int dA\, \boldsymbol{\zeta} \cdot \hat{\mathbf{n}} = \oint \mathbf{u} \cdot d\mathbf{x}$$

The vorticity is directly related to the circulation; we can think of it as the local spin or swirl of the fluid.

Because of the Earth's rotation, the flows we deal with are inherently rotational, meaning $\boldsymbol{\zeta} \neq 0$. To see this, consider the motion of a particle which is stationary with respect to the Earth. Viewed in an inertial frame, though, it is moving in a circular path with

$$u = \Omega r \quad \Rightarrow \quad \oint \mathbf{u} \cdot d\mathbf{x} = 2\pi\Omega r^2 \quad \Rightarrow \quad \zeta = 2\pi\Omega r^2/\pi r^2 = 2\Omega$$

Since large-scale flows are dominated by horizontal motions, we need to consider the circulation on the surface of the spherical earth.

$$\oint \mathbf{u} \cdot d\mathbf{x} = \Omega a^2 [\cos^2(\theta - d\theta/2) - \cos^2(\theta + d\theta/2)]d\lambda$$

$$= -\Omega a^2\, d\lambda d\theta\, \frac{d}{d\theta} \cos^2\theta$$

$$= 2\Omega a^2\, d\lambda d\theta\, \cos\theta \sin\theta$$

$$\int \boldsymbol{\zeta} \cdot dA = \zeta_3\, a\cos\theta d\lambda\, a d\theta$$

$$\Rightarrow \zeta_3 = 2\Omega \sin\theta$$

This is known as the Coriolis parameter: $f = 2\Omega \sin\theta$.

The identification of vorticity with rotation makes the vorticity-Bernoulli form of the equations

$$\frac{\partial}{\partial t}\mathbf{u} + (\boldsymbol{\zeta} + 2\boldsymbol{\Omega}) \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\frac{1}{2}|\mathbf{u}|^2 - \nu\nabla\times\boldsymbol{\zeta}$$

more intuitive.

Flows generally have both vorticity and strain; for example a simple shear $u = sy$ has $\zeta_3 = -s$ and $S_{12} = S_{21} = \frac{1}{2}s$. (An exception is solid body rotation $u = -\omega y$, $v = \omega x$ which has vorticity 2ω but $\mathbf{S} = 0$: blobs of fluid rotate but do not stretch out.)

trajectories

Conservation properties [nondivergent flow]

If we have a quantity satisfying

$$\frac{\partial}{\partial t}b + \mathbf{u}\cdot\nabla b = 0 \quad or \quad \frac{\partial}{\partial t}b + \nabla\cdot\mathbf{u}b = 0$$

and we integrate over the fluid volume

$$\frac{\partial}{\partial t}\int dV\, b = -\int dS\,\mathbf{u}\cdot\hat{\mathbf{n}}\,b = 0$$

Now suppose we have a local maximum in b at some point $\mathbf{x}$. Apply the same argument to a volume whose surface is defined by $b = b_0$; then

$$\frac{\partial}{\partial t}\int dV\, b = -\int dS\,\mathbf{u}\cdot\hat{\mathbf{n}}\,b_0 = -b_0\int dS\,\mathbf{u}\cdot\hat{\mathbf{n}} = -b_0\int dV\,\nabla\cdot\mathbf{u} = 0$$

Thus the peak value is preserved. Another way to see this is to consider Lagrangian variables:

$$b(\mathbf{X}(t), t) = b(\mathbf{X}(0), 0) \quad \text{where} \quad \frac{\partial}{\partial t}\mathbf{X}(t) = \mathbf{u}(\mathbf{X}(t), t)$$

Basics of diffusion

Diffusion, on the other hand, reduces maxima and tries to make b uniform. Consider

$$\frac{\partial}{\partial t}b = \nabla\cdot\kappa\nabla b \quad \Rightarrow \quad \frac{\partial}{\partial t}\int dV\, b = \int dS\,\kappa\,\hat{\mathbf{n}}\cdot\nabla b$$

Over the entire volume, this vanishes with no-flux conditions. Around a local maximum, however, the r.h.s. is negative definite implying the peak value will decrease. If we split b into the volume average $\bar{b}$ and the deviations, we have that

$$\frac{\partial}{\partial t}\langle b\rangle = 0 \quad , \quad \frac{\partial}{\partial t}\int dV\,\frac{1}{2}b'^2 = -\int\kappa|\nabla b'|^2$$

So that the variance will steadily decrease.

Conservation, homogenization, and movement

Currency-based reaction-advection-diffusion models in which motion and mixing are purely physical share one property: no matter how many compartments or how complex the interactions are, the total

$$b_T = \sum_i b_i$$

satisfies the same advection-diffusion equation as a passive tracer. By construction, we have accounted for all the exchanges of, say, nitrogen so that

$$\sum_i \mathcal{B}_i = 0 \quad ;$$

summing the individual biological equations gives

$$\frac{\partial}{\partial t} b_T + \nabla \cdot \mathbf{u}_i b_i = \nabla \cdot \kappa_i \nabla b_i$$

If no biological movement or dispersion occurs, then

$$\frac{\partial}{\partial t} b_T + \mathbf{u} \cdot \nabla b_T = \nabla \cdot \kappa \nabla b_T$$

which has important consequences:
- Local maxima or minima are smoothed out with time.
- If there is no flux of material through the boundaries, b_T will become spatially and temporally uniform.
- If the values of b_T at the boundaries are held fixed, the maximum and minimum values will be found on the boundary.

We've discussed the first and second; diffusion plus the fact that advection cannot creat local maxima or minima gives the third.

The implications of the second statement in the deep ocean are striking: almost all of b_T will be in the dissolved forms (e.g. nitrate plus nitrite) and the total of those would be uniform. Since observations indicate significant variability with large and systematic gradients, we cannot expect a closed, purely advective/diffusive model to work well for an ocean basin.

Inhomogeneous sources or sinks for one or more constituents (e.g., riverine input of nutrients) can lead to inhomogeneous distributions for the same reasons that salinity is nonuniform – the variations on the top or sides will be advected into the interior and their amplitude will be attenuated, but not erased, by diffusion.

In addition, living creatures (or, for that matter, detrital material) move relative to the water so the $\mathbf{u}_i$ and κ_i can be different from the fluid motion. In that case

$$\frac{\partial}{\partial t} \langle b_T \rangle = 0$$

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$$\frac{\partial}{\partial t}b'_T + \nabla \cdot (\mathbf{u}_i b'_i - \kappa_i \nabla b'_i) = 0$$

$$\frac{\partial}{\partial t}\frac{1}{2}b'^2_T + b'_i \nabla \cdot (\mathbf{u}_i b'_i - \kappa_i \nabla b'_i) = 0$$

$$\frac{\partial}{\partial t}\langle\frac{1}{2}b'^2_T\rangle = -\langle\frac{1}{2}b'^2_i \nabla \cdot \mathbf{u}_i\rangle - \langle\kappa_i|\nabla b'_i|^2\rangle$$

Variance decreases because of diffusion, but can increase when some of the biological velocities are convergent.

Stability

Stability generally deals with the following question: if we have one solution to the equations (called the basic state), will a perturbed solution which is initially near the basic state remain near it? To assess this with a given model, we need to define the basic state $\bar{b}_i$ and then solve the equation for the perturbation $b'_i = b_i - \bar{b}_i$

$$\frac{\partial}{\partial t}b'_i + \nabla \cdot \mathbf{u}_i b'_i - \nabla \cdot \kappa_i \nabla b'_i = \mathcal{B}_i(\bar{b} + b') - \mathcal{B}_i(\bar{b})$$

(This ignores density dependence in the biological velocities and diffusivities). We want to know whether all solutions with $||b'(0)||$ small will have $||b'(t)||$ remaining small. During the phase when the perturbations are small, we can solve for their evolution by linearizing the right side

$$\mathcal{B}_i(\bar{b} + b') - \mathcal{B}_i(\bar{b}) \simeq \frac{\partial \mathcal{B}_i}{\partial b_j}b'_j = \mathcal{B}_{ij}b'_j$$

(summed over j). The $\mathcal{B}_{ij}$ matrix is evaluated at the local basic state values.

Mostly we deal with basics states which are steady neglecting time dependence in $\mathcal{B}$. In that case, $\mathcal{B}_{ij}$ is a constant matrix; the pure biological problem

$$\frac{\partial}{\partial t}b'_i = \mathcal{B}_{ij}b'_j$$

has solutions of the form

$$b'_i(t) = \exp(t\mathcal{B}_{ij})\,b'_j(0)$$

where the matrix exponential can be defined by a Taylor series

$$\exp(t\mathcal{B}_{ij}) = \frac{1}{n!}t^n\mathcal{B}^n_{ij}$$

but more usefully in terms of the eigenvalues and eigenvectors of $\mathcal{B}_{ij}$:

$$\mathcal{B}_{ij}v_{jk} = v_{ik}\sigma_k$$

Here the kth column of v_{ik} is the eigenvector corresponding to eigenvalue σ_k. Then

$$\exp(t\mathcal{B}_{ij}) = v^{-1}_{ik}e^{\sigma_k t}v_{kj}$$

When the real part of every eigenvalue is negative, all perturbations will eventually decay and we call the equilibrium stable; if the real part of one or more is positive, that mode will grow to large amplitude (when the Taylor approximation breaks down) and we call the equilibrium unstable. Purely imaginary eigenvalues correspond to neutral modes which simply oscillate; the linearized equations are not entirely adequate. As we shall discuss later, the transient behavior even of a stable system can show considerable temporary growth, so that an asymptotic stability result may be misleading, but it is at least a start.

For the Q-NPZ model, the matrix $\mathcal{B}_{ij}$ is

$$\mathcal{B}_{ij} = \begin{pmatrix} \mu(N_T - \overline{P} - \overline{Z}) - g\overline{Z} - d_P - \mu\overline{P} & -(\mu + g)\overline{P} \\ ag\overline{Z} & ag\overline{P} - d_Z \end{pmatrix}$$

which simplifies for the three equilibria to

$$\overline{P} = \overline{Z} = 0 \quad , \quad \mathcal{B}_{ij} = \begin{pmatrix} \mu N_T - d_P & 0 \\ 0 & -d_Z \end{pmatrix}$$

$$\overline{Z} = 0, \ \overline{P} = N_T - d_P/\mu \quad , \quad \mathcal{B}_{ij} = \begin{pmatrix} -\mu\overline{P} & -(\mu + g)\overline{P} \\ 0 & ag\overline{P} - d_Z \end{pmatrix}$$

$$\overline{P} = d_z/ag, \ \overline{Z} = (\mu N_T - \mu d_Z/ag - d_P)/(\mu + g) \quad , \quad \mathcal{B}_{ij} = \begin{pmatrix} -\mu\overline{P} & -(\mu + g)\overline{P} \\ ag\overline{Z} & 0 \end{pmatrix}$$

For a 2×2 real matrix, at least one of the eigenvalues will have a positive real part if the trace (Tr, the sum of the diagonal elements) is positive or the determinant (Det, the product of the diagonal elements minus the product of the off-diagonal elements) is negative. Thus the $\overline{P} = \overline{Z} = 0$ state becomes unstable when $\mu N_T > d_P$ – phytoplankton have enough nutrient to grow. The $\overline{P} \neq 0$, $\overline{Z} = 0$ state will become unstable when $ag\overline{P} > d_Z$; at this point, the phytoplankton have sufficient biomass to support zooplankton growth. In both cases, the determinant switches from positive to negative before the trace becomes positive. Finally, the state with both $\overline{P}$ and $\overline{Z}$ non-zero will always have negative Tr and positive Det, so that all perturbations will decay .

For the NPZ(M) [Michaelis-Menten/ Monod] model, it is useful to pose the problem in terms of *per-capita* rates

$$\frac{\partial}{\partial t}b_i = b_i \mathcal{R}_i(b)$$

so that steady states have either $b_i = 0$ or $\mathcal{R}_i = 0$. The perturbation problem

$$\frac{\partial}{\partial t}b_i' = \mathcal{R}_i(\overline{b})b_i' + \overline{b}_i\frac{\partial \mathcal{R}_i}{\partial b_j}b_j'$$

can have growth if $\overline{b}_i = 0$ and $\mathcal{R}_i > 0$ or if $\overline{b}_i \neq 0$, $\mathcal{R}_i = 0$, and at least one eignevalue of $\mathcal{R}_{ij}$ is positive. For a system like the Franks (2002) formulation

$$\mathcal{R}_p = I(light)u(N) - g(P)Z - d_p(P)$$
$$\mathcal{R}_z = aPg(P) - d_z(Z)$$

and

$$\mathcal{B}_{ij} = \begin{pmatrix} \overline{P}(-Iu' - g'\overline{Z} - d'_p) & -\overline{P}(Iu' + g) \\ \overline{Z}a(Pg)' & -\overline{Z}d'_z \end{pmatrix}$$

(primes being derivatives). With type II curves, $u' > 0$, $(Pg)' > 0$ but $g' < 0$. If $d'_z = 0$ (linear mortality on Z), Det will be positive, and instability can only occur if $-Iu' - d'_p - g'\overline{Z} > 0$. For the NPZ(M) form, we need

$$\frac{g\overline{Z}}{(\overline{P} + P_{1/2})^2} > \frac{\mu N_{1/2}}{(\overline{N} + N_{1/2})^2}$$

Biological terms dominant

Sometimes we can think of the physics as weakly perturbing the biological state; if

$$\frac{\partial}{\partial t}\overline{b}_i = \mathcal{B}_i(\overline{\mathbf{b}}, \mathbf{x}, t)$$

then

$$\frac{\partial}{\partial t}b'_i \simeq \mathcal{B}_{ij}b'_j - \nabla \cdot (\mathbf{u}_i\overline{b}_i - \kappa_i\nabla\overline{b}_i)$$

When $\mathcal{B}$ depends only on space, not time, this can be solved formally as

$$\frac{\partial}{\partial t}b'_i = e^{t\mathcal{B}_{ij}}b'_j(0) - \int_0^t dt' e^{(t-t')\mathcal{B}_{ij}}\nabla \cdot [\mathbf{u}_j(t')\overline{b}_j(t') - \kappa_j\nabla\overline{b}_j(t')]$$

If the rates associated with $\mathcal{B}_{ij}$ (the eigenvalues) are large,

$$\mathcal{B}_{ij}b'_j = \nabla \cdot (\mathbf{u}_i\overline{b}_i - \kappa_i\nabla\overline{b}_i)$$

Note the cross connections here; in the NPZ case, we have

$$\frac{agP_{1/2}}{(P_{1/2} + \overline{P})^2}P' = \nabla \cdot (\mathbf{u}_i\overline{Z} - \kappa_i\nabla\overline{Z})$$

Advection and diffusion of Z gradients cause perturbations in the P biomass.

A similar analysis can be done assuming the basic state includes diffusion but not advection; the forcing term then is the $\mathbf{u} \cdot \nabla\overline{b}$. In the equivalent of Franks' problem, $\overline{b}$ is a function only of z and

$$\mathcal{B}_{ij}b'_j + \nabla \cdot \kappa\nabla b'_i = w\frac{\partial}{\partial z}\overline{b}_i$$

The response to upwelling(ignoring κ) show an increase in P and loss of Z in the deeper water. The two-d calculation shows the development of a deep bloom as in Franks' study; this experiment starts from rest however and takes about two years to settle. The movie shows a longer development, and, at the end, the upwelling, center, and downwelling areas resemble the prediction above, but not that closely.

Small amplitude characteristics

For a number of problems, we can learn a lot by looking at the behavior near the origin (for the biotic variables)

$$\frac{D}{Dt}b_i \simeq \mathcal{R}_i\big|_0 b_i + \nabla \cdot \kappa \nabla b_i$$

or

$$\frac{D}{Dt}b = g(\mathbf{x})b + \nabla \cdot \kappa \nabla b$$

where we'll use g for $\mathcal{R}_i\big|_0$ and drop the i subscript.

General comments

We can divide the equation by b and derive the one for $\beta = \ln(b)$:

$$\frac{D}{Dt}\beta = g(\mathbf{x}) + \nabla \cdot \kappa \nabla \beta + \kappa |\nabla \beta|^2$$

If β is growing, and we look near a local maximum, the g term must be big enough to overcome the diffusive losses from the 2nd term. The 3rd term will be higher order, as will be the curvature terms in g.

Moving fronts

Take $\mathbf{u} = 0$ and $g = const.$; there are solutions like

$$\exp(-k[x - ct])$$

with

$$kc = g + \kappa k^2$$

The minimum c value occurs at

$$k = \sqrt{g/\kappa} \quad , \quad c = 2\sqrt{g\kappa}$$

The front develops exponentially at the leading edge and then (in the presence of a logistic term) equilibrates. The speed becomes constant (as in the 3D plot). With an NPZ model we see two fronts. In general, these correspond to solutions coming out of the unstable point at the origin and then connecting to some nonlinear equilibrium.

Local growth

Suppose we have a spatially variable growth rate which is negative over most of the domain, with regions of positive values. When can a starter population grow? We can look for exponential solutions

$$\sigma b = g(\mathbf{x})b + \nabla \cdot \kappa \nabla b$$

This is a form of Schrödinger's eqn; we are looking for "bound states" with b positive. If $\sigma > 0$, this solution will grow and establish a resident, self-sustaining population.

We can roughly describe the effects of nonlinearity if it's not too strong. For a logistic system

$$\frac{\partial}{\partial t} b = g(\mathbf{x})b - h(\mathbf{x})b^2 + \nabla \cdot \kappa \nabla b$$

suppose we have a neutral solution

$$0 = \alpha g(x)b_0 + \nabla \cdot \kappa \nabla b_0$$

with $\alpha < 1$. Then we have

$$\frac{\partial}{\partial t}\langle b_0 b \rangle = \langle g b_0 b \rangle + \langle b_0 \nabla \cdot \kappa \nabla b \rangle - \langle h b_0 b^2 \rangle = \langle g b_0 b \rangle - \alpha \langle g b_0 b \rangle - \langle h b_0 b^2 \rangle$$

If we approximate $b \sim A(t)b_0(\mathbf{x})$, we have

$$\langle b_0^2 \rangle \frac{\partial}{\partial t} A = (1 - \alpha)\langle g b_0^2 \rangle A - \langle h b_0^3 \rangle A^2$$

so that the amplitude grows and saturates.

As an example, consider a circular bank of radius ℓ with positive growth rate g_0 surrounded by an infinite area with negative growth rate g_1. We can guess that survival will rely on the growth time $1/g_0$ being less that the time for diffusive loss ℓ^2/κ or $g_0 \ell^2/\kappa > 1$, but with some number other than 1. We can find the interior and exterior solutions

$$b = \begin{cases} J_0(k\frac{r}{\ell})K_0(k') & r < \ell \\ J_0(k)K_0(k'\frac{r}{\ell}) & r > \ell \end{cases}$$

Matching the derivatives implies a relation between k and k'

$$kJ_1(k)K_0(k') = k'J_0(k)K_1(k')$$

The seed population increases at a rate

$$\sigma = g_0 - \frac{\kappa}{\ell^2}k^2 = g_1 + \frac{\kappa}{\ell^2}k'^2$$

This gives a second criterion

$$g_0 - g_1 = \frac{\kappa}{\ell^2}(k^2 + k'^2)$$

which determines the value of k. For the solution to grow, we require

$$\frac{g_0 \ell^2}{\kappa} > k^2 \quad \text{and} \quad \frac{g_1 \ell^2}{\kappa} > -k'^2$$

For the example shown, $k = 1.75$ and $k' = 2.29$; if we take $g_1 \ell^2/\kappa = -3$ and $g_0 \ell^2/\kappa = 5.31$, than $\sigma \ell^2/\kappa = 2.25$ (shape of soln). In any case, the estimate above is ok, but the number is more like 4.

Washout

Now let us look at a case with flow through a region of positive growth. Again, a self-sustaining population has

$$\sigma b + \mathbf{u} \cdot \nabla b = g(\mathbf{x})b + \nabla \cdot \kappa \nabla b$$

with $\sigma > 0$. Dimensions: U $[L/T]$, g $[1/T]$, ℓ $[L]$, and κ $[L^2/T]$ 4 quantities and two dimensions implying two non-dim parameters.

$$\frac{g\ell^2}{\kappa} > \text{func}(\frac{U\ell}{\kappa}) = \text{const} + \frac{1}{4}Pe^2$$

We can look at the 1D problem for insight; let $\mathbf{u} = U\hat{\mathbf{x}}$ so that

$$\sigma b + U\frac{\partial}{\partial x}b = g(x)b + \kappa\frac{\partial^2}{\partial x^2}b$$

We take
$$b = f(x)\exp(Ux/2\kappa)$$

to find
$$\sigma f = \left[g(x) - \frac{U^2}{4\kappa}\right]f + \kappa\frac{\partial^2}{\partial x^2}f$$

The condition for growth now becomes

$$g_0 - \frac{U^2}{4\kappa} > \text{const}.\frac{\kappa}{\ell^2}$$

Thus we need U to be sufficiently less than the invasion speed $2\sqrt{g_0\kappa}$; the population must be able to return to the growing region by diffusion aided by growth faster than it is washed out by the flow.

In the 1D problem, the population fails if κ is too small or too big. In 2D, if the favorable region is a bank, the flow may go around to some extent with closed streamlines on the peak. In that case, small diffusivity may not be such a problem.

We cannot eliminate the flow in the same way; in 2D with $\mathbf{u} = \hat{\mathbf{z}} \times \nabla\psi - \nabla\phi$, we can set $b = f\exp(-\phi/2\kappa)$ to find

$$\frac{\partial}{\partial t}f + \hat{\mathbf{z}} \cdot \nabla\psi \times \nabla f = \tilde{g}f + \kappa\nabla^2 f$$

$$\tilde{g} = g + \hat{\mathbf{z}} \cdot \nabla\psi \times \nabla\phi/2\kappa - |\nabla\phi|^2/4\kappa + \frac{1}{2}\nabla^2\phi$$

Advection dominant

As an example of a case with advection dominant, consider the problem above and integrate over the area enclosed by a streamline

$$\frac{\partial}{\partial t}\int_A f + \oint_{\partial A}\mathbf{u}\cdot\hat{\mathbf{n}}f = \int_A \tilde{g}f + \kappa\oint\hat{\mathbf{n}}\cdot\nabla f$$

If ∂A is a streamline, the $\mathbf{u}$ is parallel to the boundary, and the second term vanishes. At lowest order, $f = F(\psi)$ and

$$\frac{\partial}{\partial t}\int_A f = \int_A \tilde{g}f + F'\kappa\oint|\nabla\psi|$$

Differentiating with respect to ψ gives

$$\frac{\partial F}{\partial t}\oint\frac{d\ell}{|\nabla\psi|} = F\oint\frac{\tilde{g}d\ell}{|\nabla\psi|} + \frac{\partial}{\partial\psi}\left[\kappa\oint|\nabla\psi|d\ell\right]\frac{\partial}{\partial\psi}F$$

So we expect the variables to become homogenized alsong streamines and then the differences from streamline to streamline will be smoothed out by diffusion.

Example – flow around a bank

We want to consider whether a population which can only grow in a limited region can survive against advective and diffusive losses. Georges Bank, for example, has higher densities in many species compared to the Gulf of Maine or the Shelf Water. Are these locally retained populations, or are they supported by import from elsewhere? We shall try to formulate this kind of problem in a crude gulf/ bank geometry with the assumption that the biological processes favor growth on the bank.

For simplicity, we consider a logistic model

$$\frac{\partial}{\partial t}b + \mathbf{u}\cdot\nabla b - \nabla\cdot\kappa\nabla b = \begin{cases} gb\left(1 - \dfrac{b}{b_0(x,y)}\right) - db & b < b_0 \\ -db & b \geq b_0 \end{cases}$$

The carrying capacity (related to the density of the resource) is presumed to vary spatially and to be localized. For example, we could consider organisms that can only reproduce in shallow enough water.

In the absence of flow, the population is simply

$$b = b_0(x,y)\left(1 - \frac{d}{g}\right)$$

so that we will deal with a problem where the carrying capacity is zero outside the home range.

Alternatively, we can examine the case where the growth rate varies spatially (vertically averaged light level, for example) giving

$$\frac{\partial}{\partial t}b + \mathbf{u} \cdot \nabla b - \nabla \cdot \kappa \nabla b = \begin{cases} g(x,y)\, b \left(1 - \dfrac{b}{b_0}\right) - db & b < b_0 \\ -db & b \geq b_0 \end{cases}$$

which has steady solutions

$$b = \begin{cases} b_0 \left(1 - \dfrac{d}{g(x,y)}\right) & g(x,y) > d \\ 0 & g(x,y) \leq d \end{cases}$$

The second form is a bit easier to deal with; if $b < b_0$ initially, it should never exceed the carrying capacity so that you don't need to test for that. (Numerics may violate this condition, but the logistic form still leads b to decay back to b_0, so it's not a serious issue.)

Flow field

For the flow, we neglect effects of stratification and friction, and we idealize the geometry to a domain with walls on the north and west, a circular bank

$$Topog = h_0 J_0(kr)$$

and a uniform flow from the east impinging on the domain.

Note that the hydrostatic equations

$$\frac{D}{Dt}u - fv = -\frac{\partial}{\partial x}\frac{p}{\rho}$$

$$\frac{D}{Dt}v + fu = -\frac{\partial}{\partial y}\frac{p}{\rho}$$

$$\frac{\partial}{\partial z}\frac{p}{\rho} = -g$$

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$$

can be solved in a consistent fashion with

$$\frac{\partial}{\partial z}u = \frac{\partial}{\partial z}v = 0$$

since $\frac{\partial}{\partial z}[\frac{\partial}{\partial x}(p/\rho)]$ is zero. Thus neither the pressure terms nor the Coriolis terms in the momentum equations will cause z structure to develop. Likewise, in the material derivative,

$$\frac{D}{Dt}u = \frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + (w\frac{\partial}{\partial z}u)$$

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the last term will vanish and the other terms have no z-dependence.

For this barotropic flow, the vertical velocity satisfies

$$\frac{\partial^2}{\partial z^2}w = 0 \quad \Rightarrow \quad w = w_0(x, y) + w_1(x, y)z$$

Applying the boundary conditions $w = 0$ at the top ($z = 0$) and the impenetrability of the bottom

$$w = (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})Topog \quad at \quad z = -H + Topog \equiv -h(x, y)$$

gives

$$w = \frac{z}{h}\mathbf{u}_h \cdot \nabla h$$

(with $\mathbf{u}_h$ being the horizontal velocity) and

$$\nabla \cdot \mathbf{u}_h + \frac{1}{h}\mathbf{u}_h \cdot \nabla h = 0 \quad or \quad \nabla \cdot (\mathbf{u}_h h) = 0$$

From this equation, we see that the flow can be defined in terms of a transport stream-function

$$hu = -H\frac{\partial}{\partial y}\psi \quad , \quad hv = H\frac{\partial}{\partial x}\psi$$

To predict the flow, we find an equation for the "potential vorticity." We write an equation for the vertical component of the vorticity

$$\zeta = \hat{\mathbf{z}} \cdot (\nabla \times \mathbf{u}) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Take an x derivative of the $\frac{\partial v}{\partial t}$ equation and subtract a y derivative of the $\frac{\partial u}{\partial t}$ equation:

$$\frac{D}{Dt}(\zeta + f) + (\zeta + f)\nabla \cdot \mathbf{u}_h = 0$$

Combining this with the conservation of mass in a column gives

$$\frac{D}{Dt}(\zeta + f) - \frac{\zeta + f}{h}\frac{D}{Dt}h = 0 \quad \Rightarrow \quad \frac{D}{Dt}\frac{\zeta + f}{h} = 0$$

Thus the potential vorticity, $q \equiv \frac{\zeta + f}{h}$ is conserved. We can express this as

$$q = \frac{\nabla \cdot \frac{H}{h}\nabla\psi + f}{h}$$

and our dynamical equation

$$\frac{\partial}{\partial t}q + \frac{H}{h}\left(\frac{\partial \psi}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial q}{\partial x}\right) = 0$$

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allows us to predict ψ.

When the flow is steady, fluid moves along the streamlines and the value of any conserved quantity is uniform on a streamline. Thus

$$q = Q(\psi)$$

If we evaluate this where the flow is coming into the domain, we can find the function Q. For uniform westward flow at $x = x_1$ and uniform depth H, we have

$$q = \frac{f}{H} = Q(-U_0 y)$$

With the "beta-plane" assumption accounting for the increase in $2\Omega \sin\theta$ with latitude — $f = f_0 + \beta y$ —, we have

$$HQ(s) = f_0 - \beta \frac{s}{U_0}$$

The dynamical equation simplifies to

$$\nabla \cdot \frac{H}{h} \nabla\psi + f_0 + \beta y = \frac{h}{H}\left(f_0 - \frac{\beta}{U_0}\psi\right)$$

or

$$\nabla \cdot \frac{H}{h} \nabla\psi = f_0 \frac{h - H}{H} - \frac{h}{H}\frac{\beta}{U_0}\psi - \beta y$$

We approximate this by replacing the h/H terms by one

$$\nabla^2 \psi = f_0 \frac{h - H}{H} - \frac{\beta}{U_0}(\psi + U_0 y)$$

This is the quasigeostrophic equation, and we can solve it for ψ.

The solutions show a population can be maintained on the bank for weak enough flow and strong enough growth.