Numerical methods

Time-stepping

Consider the Adams-Bashforth scheme described (very briefly) in class for solving

$$\frac{d}{dt}b = \mathcal{B}(b,t)$$

using

$$b_e = \alpha b(t) + (1-\alpha) b(t-dt)$$

$$b(t+dt) = b(t) + dt\, \mathcal{B}(b_e, t+\gamma dt)$$

- **Optional:** Find α and γ by ensuring that the Taylor expansions around $t=0$ of the two sides of the last equation match to at least order dt^2 and to see whether they can also match at order dt^3. First expand b_e and substitute into the last equation. Then expand away. Note that

$$\frac{d^2}{dt^2}b = \frac{d}{dt}\mathcal{B}(b,t) = \frac{\partial}{\partial t}\mathcal{B} + \frac{db}{dt}\frac{\partial}{\partial b}\mathcal{B} = \frac{\partial}{\partial t}\mathcal{B} + \mathcal{B}\frac{\partial}{\partial b}\mathcal{B}$$

  etc.

Space discretization

For an equation

$$\frac{\partial}{\partial t}b = -\frac{\partial}{\partial x}F$$

we let b_n be the average value in the grid cell $(x_0+[n-\frac{1}{2}]dx, x_0+[n+\frac{1}{2}]dx)$; you can think of it as the value at $x_0 + ndx$. We calculate the fluxes $F_{n-\frac{1}{2}}$ at the edges $x_0+[n-\frac{1}{2}]dx$. Then

$$\frac{\partial}{\partial t}b_n = \frac{1}{dx}\left[F_{n-\frac{1}{2}} - F_{n+\frac{1}{2}}\right] + \mathcal{B}(b_n, x_n, t)$$

For advective-diffusive problems,

$$F = ub - K\frac{\partial}{\partial x}b$$

becomes

$$F_{n-\frac{1}{2}} = u_{n-\frac{1}{2}}(b_{n-1}+b_n)/2 - K(b_n - b_{n-1})/dx$$

This has been implemented in a periodic domain $x=-10$ to $x=10$ with $dx = 1/8$ and constant $u=1$. The code is here.

- Start with a gaussian $b = \exp(-\frac{1}{2}x^2)$. You need dt sufficiently less than dx/u. How big does K have to be to insure positive b?

Enhancement at front

- Now modify the code to solve

$$\frac{\partial}{\partial t}b = -\frac{\partial}{\partial x}\left[ub - K\frac{\partial}{\partial x}b\right]$$

with

$$u = -x\exp(-\frac{1}{2}x^2)$$

Note that $b(1)$ is at $x(1) = -10 + dx/2$ but $u(1)$ is at -10. How large does b become compared to the value far from the front? Compare to the analytical solution $b = b_0 \exp[\frac{1}{K}\exp(-\frac{1}{2}x^2)]$. How long does it take for the peak to be pretty much established? $t = 1$ is an advective time, while $1/K$ is a diffusive time.

Searching

Now consider an individual predator searching within the field $b = b_0 \exp[\frac{1}{K}\exp(-\frac{1}{2}x^2 - \frac{1}{2}y^2)]$ now considered as a function of x and y. The predator's position is given by

$$\frac{\partial}{\partial t}X = U \quad , \quad \frac{\partial}{\partial t}Y = V$$

and the angle changes by turning towards higher values of b and accelerating up the gradient

$$\frac{\partial}{\partial t}U = [\alpha V + \beta]\frac{\partial}{\partial x}b(X,Y) \quad , \quad \frac{\partial}{\partial t}V = [-\alpha U + \beta]\frac{\partial}{\partial y}b(X,Y)$$

- Play with α and β to see what behaviors result. Use the model here. How might you change this model to give more sensible results?