## Fluid dynamics and continuum models of the biota

In fluid dynamics, we do not deal with the individual molecules bat rather are concerned with average properties such as the mass per unit volume (the density $\rho=\sum m_{i} / V$ ) and the velocity $\mathbf{u}=\sum m_{i} \mathbf{u}_{i} / \sum m_{i}$. [Bold face is used for vectors.] Likewise, for organisms, we often - but not always - consider the biomass density $b(\mathbf{x}, t)$ meaning the amount of biomass in a volume $V$ surrounding the point $\mathbf{x}$ divided by the $V$. Reaction-advectiondiffusion models are formulated as sets of PDE's giving the spatial-temporal evolution of a set of these fields.

But organisms are discrete individuals, each with its characteristics: species, physiological and physical state [position, velocity, orientation]. "Individual-based models" attempt to include some of this kind of information, but are necessarily limited to a small number of organisms. We can take several approaches to getting from individuals to a continuum representation.

- Spatial average: This is the observational approach; take a net tow filtering some volume, count the organisms, divide by the volume, and assign that density to the location of the tow. For small volumes, the statistical fluctuations make $\rho$ unreliable, while large volumes mix actual spatial variations into the estimate.
- Probabilities: This approach begins with a continuous function $\mathcal{P}(\mathbf{x}, \mathbf{u}, t)$ such that $d \mathbf{x} d \mathbf{u} \mathcal{P}$ is the probability that an organism is in a neighborhood volume $d \mathbf{x}$ around $\mathbf{x}$ and $d \mathbf{u}$ around velocity $\mathbf{u}$. We choose this form, since it also applies to the physics: molecules are characterized by their position and momentum [in classical mechanics; in quantum mech. PDF's are also the fundamental property]. We will relate changes in the PDF to the deterministic forces and the stochastic accelerations.

You can look at this pd\# for more detail; see also Chapter 2.

- Continuum dynamics: Here, we will just use the standard approach of treating the fluid and the biota as a field - a continuous function of space and time. Since we deal with macroscopic volumes, this is fine for fluid molecules (and even better with quantum mechanics), but it gets shakier as we move to scarcer organisms.

The fundamental principle in continuum dynamics for fluids is that the rate of change of the amount of some stuff $b(\mathbf{x}, t)$ in a volume around point $\mathbf{x}$ is given by the difference between the amount moving into and out of the volume and the sources and sinks. The amount leaving through a small patch of surface of the volume per unit time is given by the flux $\mathbf{F}$ dotted with the outward normal vector times the area. The flux is the amount of stuff passing through a unit area per unit time. Clearly that will depend on the orientation of the surface (described by its normal vector; the direction of $\mathbf{F}$ will be the one which maximizes this transport. For a different orientation the flux is given by $\mathbf{F} \cdot \hat{\mathbf{n}}$. Examples include advective flux, purely diffusive flux, and advection plus diffusion. -

The integral of the $\mathbf{F} \cdot \hat{\mathbf{n}}$ over the surface will tell us the net amount leaving per unit time:

$$
\frac{\partial}{\partial t} \int_{V} d \mathbf{x} b(\mathbf{x}, t)=-\oint_{S} d a \mathbf{F} \cdot \hat{\mathbf{n}}+\int_{V} d \mathbf{x} \mathcal{B}
$$

here $\mathcal{B}$ represents sources and sink. Applying the divergence theorem gives

$$
\frac{\partial}{\partial t} b=-\nabla \cdot \mathbf{F}+\mathcal{B}
$$

Mass
For mass, or density $\rho$, the flux is just $\mathbf{u} \rho$, using the fact described above that the velocities are defined by mass-weighted velocities of individual molecules.

$$
\frac{\partial}{\partial t} \rho=-\nabla \cdot(\mathbf{u} \rho)
$$

Salt
For properties measured in a per-unit-mass form, like salinity, the conservation law becomes

$$
\frac{\partial}{\partial t} \rho S=-\nabla \cdot(\mathbf{u} \rho S)
$$

However, there now can be other exchanges across the surface, in particular, diffusion which leads to a flux down the gradient. Suppose we consider the fluid to be at rest and take a small volume to the right of the surface, containing $\rho(x+d x / 2, y, z, t) S(x+$ $d x / 2, y, z, t) d x d y d z$ salt "molecules." Assume a fraction $\delta$ of these move leftward out of the box and the same fraction leave to the right. In the same way consider a box on the left (inner) side of the surface. The net amount passing the surface or area $d y d z$ is

$$
\begin{gathered}
F d y d z d t=\delta \rho(x-d x / 2, y, z, t) S(x-d x / 2, y, z, t) d x d y d z \\
-\delta \rho(x+d x / 2, y, z, t) S(x+d x / 2, y, z, t) d x d y d z
\end{gathered}
$$

Taylor expanding gives

$$
F=-\frac{\delta d x^{2}}{d t} \frac{\partial}{\partial x} \rho S
$$

or

$$
\mathbf{F}=-\kappa \frac{\partial}{\partial x} \rho S
$$

Therefore

$$
\frac{\partial}{\partial t} \rho S=-\nabla \cdot[\mathbf{u} \rho S-\kappa \nabla \rho S]
$$

Using the mass equation leads to

$$
\frac{\partial}{\partial t} S=-\mathbf{u} \cdot \nabla S+\frac{1}{\rho} \nabla \cdot(\kappa \nabla \rho S)
$$

This is usually written using the "material derivative"

$$
\frac{D}{D t} \equiv \frac{\partial}{\partial t}+\mathbf{u} \cdot \nabla
$$

and ignoring $\rho$ factors in the diffusion as

$$
\frac{D}{D t} S=\nabla \cdot(\kappa \nabla S)
$$

For salinity, the sources and sinks are at the sea surface (where they are really fluxes of water turning to vapor or from precipitation)or boundaries.

Biota
For quantities measured per-unit-volume, we just have

$$
\frac{\partial}{\partial t} b=-\nabla \cdot(\mathbf{u} b)+\nabla \cdot(\kappa \nabla b)+\mathcal{B}
$$

or

$$
\frac{D}{D t} b=-b \nabla \cdot \mathbf{u}+\nabla \cdot(\kappa \nabla b)+\mathcal{B}
$$

## Momentum

In Cartesian coordinates, we can look at individual components of the momentum per unit volume $\rho u_{i}$

$$
\frac{\partial}{\partial t} \rho u_{i}=-\nabla \cdot\left(\rho \mathbf{u} u_{i}\right)+\nabla \cdot\left(\nu \nabla \rho u_{i}\right)+F_{i}
$$

or

$$
\begin{gathered}
\frac{\partial}{\partial t} u_{i}=-\mathbf{u} \cdot \nabla u_{i}+\nu \nabla^{2} u_{i}+\frac{1}{\rho} F_{i} \\
\frac{D}{D t} u_{i}=\nu \nabla^{2} u_{i}+\frac{1}{\rho} F_{i}
\end{gathered}
$$

In non-Cartesian geometries (e.g. the spherical Earth), use

$$
\begin{gathered}
\frac{\partial}{\partial t} \rho+\operatorname{div}(\rho \mathbf{u})=0 \\
\frac{\partial}{\partial t} \mathbf{u}+\boldsymbol{\zeta} \times \mathbf{u}+\operatorname{grad}\left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u}\right)=-\nu \operatorname{curl}(\boldsymbol{\zeta})+\frac{1}{\rho} \mathbf{F} \quad, \quad \boldsymbol{\zeta}=\operatorname{curl}(\mathbf{u})
\end{gathered}
$$

and look up the appropriate forms of the curl, divergence, and gradient. This form drops the subtleties of the viscous term and the way density comes into it. Here, $\boldsymbol{\zeta}$ is the vorticity and measures the local spin of the fluid.

## Forces

The forces acting on a rotating stratified fluid are gravity (which appears as buoyancy forces), pressure, Coriolis, and viscous stresses. We need to represent each of these as the force exerted per unit volume.

Gravity: The effects of gravity are straight-forward: the force is $g$ times the mass in the downward or negative $z$ direction. The force per unit volume is

$$
\mathbf{F}=-\rho g \hat{\mathbf{z}}
$$

Coriolis "forces" act on matter moving in a rotating system. A particle moving horizontally but subject to no real horizontal forces appears to move in a curved path because the Earth is rotating under it. Consider a satellite starting over England given a push due northward in a polar orbit. It has no east-west or north-south acceleration, just gravity holding in the orbit. What does the track (marked by periodically dropped paintballs) look like relative to the ground? This animation of very slow polar orbit shows the result: the track appears curved. We ascribe this curvature to a fictitious force perpendicular to the track - the Coriolis force.


$$
\mathrm{t}=-\delta \mathrm{t}
$$

## Rotating frame



$$
t=0
$$

$$
\mathrm{t}=\delta \mathrm{t}
$$



Figure 1: Particle positions in fixed and rotating frame. Blue lines show the coordinate axes in the fixed frame, green lines in the rotating frame.

Suppose we consider three snapshots of a particle subject to no external forces viewed in both a fixed (inertial) and a rotating frame of reference. In inertial space, the particle is moving in a straight line; we set $t=0$ as the time when it passes through the origin heading along the $x$-axis. In the inertial (fixed) frame, its position is given by

$$
\mathbf{x}_{f}=\left(u_{0} t, 0,0\right)
$$

giving successive points

$$
\mathbf{x}_{f}:\left(-u_{0} \delta t, 0,0\right) \rightarrow(0,0,0) \rightarrow\left(u_{0} \delta t, 0,0\right)
$$

Correspondingly, the positions in the rotating frame are

$$
\begin{gathered}
\mathbf{x}:\left(-u_{0} \delta t \cos (\Omega \delta t),-u_{0} \delta t \sin (\Omega \delta t), 0\right) \rightarrow(0,0,0) \rightarrow \\
\left(u_{0} \delta t \cos (\Omega \delta t),-u_{0} \delta t \sin (\Omega \delta t), 0\right)
\end{gathered}
$$

where $\Omega$ is the rotation rate of the reference frame.
Clearly the particle accelerates in the $-y$ direction. Indeed, for this case, using an approximation to the second derivative gives

$$
\begin{aligned}
\frac{d^{2} \mathbf{x}}{d t^{2}} & \simeq \frac{[\mathbf{x}(t+\delta t)-\mathbf{x}(t)]-[\mathbf{x}(t)-\mathbf{x}(t-\delta t)]}{\delta t^{2}} \\
& =\frac{\mathbf{x}(t+\delta t)+\mathbf{x}(t-\delta t)-2 \mathbf{x}(t)}{\delta t^{2}} \\
& =\frac{\left(0,-2 u_{0} \delta t \sin (\Omega \delta t), 0\right)}{\delta t^{2}} \\
& =-2 \Omega u_{0} \hat{\mathbf{y}}=-2 \vec{\Omega} \times \mathbf{u}
\end{aligned}
$$

Applying the same argument to a particle moving north shows that it also accelerates to the right.

If we were to postulate some force as causing this acceleration, the strength would be

$$
\mathbf{F}=-\rho 2 \vec{\Omega} \times \mathbf{u}
$$

This Coriolis "force" is of course an artifact of dealing with movement in an accelerating reference frame (remember that circular motion has a velocity vector which is constantly changing with time) but it can be used just as though it were real. Usually, however, we will put this term on the left-hand side to keep it with the accelerations relative to the earth

$$
\frac{D}{D t} \mathbf{u}+2 \vec{\Omega} \times \mathbf{u}=-g \hat{\mathbf{z}}+\frac{1}{\rho} \mathbf{F}
$$

with $F$ being the remaining two forces.
Pressure represents the forces that the molecules exert as they bounce off each other during their thermal fluctuations (not the average velocity $\mathbf{u}$ ). Conceptually, if we consider a wall in a fluid with no average motion, each time a molecule bounces off a wall, it applies a force to the wall (and the wall applies an equal and opposite force to reverse the normal component of the molecule's velocity). The net force is the product of the average normal velocity, the mass of the molecules, and the number hitting the wall per unit time. If we double the size of the wall, we double the number of molecules impinging on it, and double the force. To account for this, we define the pressure as the force per unit area.

Now consider the forces on a small cube-shaped object centered on location $x$ in the fluid. If the thermal motion is the same everywhere in the fluid, the forces exerted on the box by molecules bouncing off the left wall will be equal and opposite to that exerted by molecules bouncing off the right wall. Therefore the net force on the cube will be zero. But if the speeds of the molecules on the right are higher than that of those on the left,
the force on the right side of the box pushing it to the left will be greater than the force on the left side pushing it to the right. The non-zero net force depends on changes in pressure and will try to push the box towards the lower pressure regions. The same argument applies if we replace the solid box with a parcel of fluid; if the molecules on the right are moving faster, collisions with them will apply more force on the fluid parcel than those with the molecules on the left. Thus we can see that the force depends on the gradient of the pressure.

To formalize this, we use the definition of pressure, as the normal force per unit area exerted by fluid outside a volume on the fluid inside, to write

$$
\mathbf{F} V=\int_{\partial V}-p \hat{\mathbf{n}} d^{2} \mathbf{x} \quad \Rightarrow \quad F_{1}=-\frac{1}{V} \int_{\partial V} p \hat{\mathbf{x}} \cdot \hat{\mathbf{n}} d^{2} \mathbf{x}=-\frac{1}{V} \int_{V} \nabla \cdot(\hat{\mathbf{x}} p) d^{3} \mathbf{x}
$$

In the limit, the force per unit volume is

$$
F_{1}=-\frac{\partial p}{\partial x} \quad \Rightarrow \quad \mathbf{F}=-\nabla p
$$

Viscous stresses are tangential forces acting across a surface; conceptually, a faster moving (on average) eastward stream located (for example) to the north of a slower stream will impart some of its momentum to the slower stream by collisions between the molecules, in effect exerting an eastward force. The slower stream has the opposite effect on the faster one. Thus, the tendency is to equalize the velocities; the stresses act much like diffusion of velocity

$$
\mathbf{F}=\rho \nu \nabla^{2} \mathbf{u}
$$

where $\nu$ is the kinematic viscosity having units (like diffusivity) of $L^{2} / T$.
Momentum equations: Putting all the forces together gives the momentum equations

$$
\begin{equation*}
\frac{D}{D t} \mathbf{u}+2 \vec{\Omega} \times \mathbf{u}=-\frac{1}{\rho} \nabla p+\nu \nabla^{2} \mathbf{u}-g \hat{\mathbf{z}} \tag{1}
\end{equation*}
$$

## Thermodynamics, buoyancy, and the Boussinesq approximation

The momentum and mass equations are not sufficient to predict the evolution of the flow: given the current state at time $t$, we know how $\mathbf{u}$ and $\rho$ change with time but cannot determine $p$ at $t+\delta t$. Fluids have an equation of state relating the density to other properties including the pressure; for seawater, this is expressed as

$$
\rho=\rho(S, T, p)
$$

where $S$ is the salinity (grams of salt per kilogram of seawater) and $T$ is the temperature. If $\rho$ were only a function of pressure, we could invert the relationship to find the new pressure given the new density; however, the dependence on $T$ and $S$ implies we need two additional evolution equations.

For simplicity, we shall avoid these complications and make the Boussinesq approximation. We let

$$
\rho \equiv \rho_{0}(z)(1-\alpha \theta)
$$

The quantity $\alpha g \theta=g \frac{\rho_{0}-\rho}{\rho_{0}}$ represents the buoyant acceleration, upwards when the density is lower than average and downwards when it is higher; in the fluid, the effects of gravity are much reduced - most of it is compensated for by pressure forces. The $\rho_{0}(z)$ takes into account the most significant part of the compressibility of sea water, the overall increase in density with depth. If we treat salt and heat separately, then we'd use $\alpha \theta=\alpha T-\beta S$; people also use a full equation of state $\rho\left(T_{p o t}, S, p\right)$ as well.


Figure 2: The net force per unit mass is $\left(-p_{0} A+p_{0} A+\rho_{0} g h A-\rho g h A\right) / \rho h A=$ $g\left(\rho_{0}-\rho\right) / \rho$.

We also define a pressure-like quantity $\phi$ such that

$$
p=-\int^{z} \rho_{0} g+\rho_{0} \phi
$$

so that the pressure gradient and gravitational terms become

$$
\begin{aligned}
-\frac{1}{\rho} \nabla p-g \hat{\mathbf{z}} & =-\frac{1}{1-\alpha \theta} \nabla \phi+g\left(\frac{1}{1-\alpha \theta}-1\right) \hat{\mathbf{z}}-\frac{\phi}{1-\alpha \theta} \frac{1}{\rho_{0}} \frac{\partial \rho_{0}}{\partial z} \hat{\mathbf{z}} \\
& =-\frac{1}{1-\alpha \theta} \nabla \phi+\frac{\alpha g \theta}{1-\alpha \theta} \hat{\mathbf{z}}-\frac{\phi}{1-\alpha \theta} \frac{1}{\rho_{0}} \frac{\partial \rho_{0}}{\partial z} \hat{\mathbf{z}} \\
& \simeq-\nabla \phi+\alpha g \theta \hat{\mathbf{z}}
\end{aligned}
$$

where the last step assumes that $\alpha \theta$ and $N^{2} H / g$ are small.
The thermodynamic and salinity equations give

$$
\frac{D}{D t} \theta=\kappa \nabla^{2} \theta+\mathcal{H}
$$

where $\mathcal{H}$ represents buoyancy sources from heating or freshening. We've assumed that (1) both the flow speed and $\sqrt{g H}$ (the long surface wave speed) are small compared to the sound speed and (2) $\kappa$ represents small scale mixing which transfers heat and salt similarly rather than the molecular processes which give quite different diffusivities.

Neglecting terms of similar order in the mass conservation equation show that the flow is nearly non-divergent. Putting these equations together gives the Boussinesq system:

$$
\begin{align*}
\frac{D}{D t} \mathbf{u}+2 \vec{\Omega} \times \mathbf{u} & =-\nabla \phi+\alpha g \theta \hat{\mathbf{z}}+\nu \nabla^{2} \mathbf{u} \\
\nabla \cdot \mathbf{u} & =0  \tag{Bouss}\\
\frac{D}{D t} \theta & =\kappa \nabla^{2} \theta+\mathcal{H}
\end{align*}
$$

## Primitive equations

For many kinds of motion, the horizontal scale is much larger than the vertical. By "scale" we mean the estimated variance in a field divided by the variance of the gradient $\left|\frac{\partial \phi}{\partial x}\right| \sim \frac{1}{L}|\phi|$. We can estimate the sizes of terms in the continuity (mass) equation as

$$
\begin{aligned}
& \frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \\
& \frac{U}{L}=\frac{W}{H}
\end{aligned}
$$

Since we don't expect the flow to be independent of $x$, the vertical velocity will be order $W=U H / L$ and will be small if the horizontal scale $L$ is much larger than the vertical scale $H$. The horizontal momentum equation, with time scale order $L / U$, has sizes like

$$
\begin{aligned}
& \frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+w \frac{\partial u}{\partial z}=-\frac{\partial \phi}{\partial x} \\
& \frac{U^{2}}{L} \quad \frac{U^{2}}{L} \quad \frac{U H}{L} \frac{U}{H} \quad \frac{\Phi}{L}
\end{aligned}
$$

The pressure will scale like $\Phi \sim U^{2}$. The vertical momentum is a different story

$$
\begin{gathered}
\frac{\partial w}{\partial t}+u \frac{\partial w}{\partial x}+w \frac{\partial w}{\partial z}=-\frac{\partial \phi}{\partial z}+\alpha g \theta \\
\frac{U^{2} H}{L^{2}} \quad \frac{U^{2} H}{L^{2}} \quad \frac{U^{2} H}{L^{2}} \quad \frac{U^{2}}{H} ?
\end{gathered}
$$

The acceleration terms are order $H^{2} / L^{2}$ smaller than the pressure gradient, and the density anomalies will scale like $U^{2} / g H$ which is also generally small $(\sqrt{g H}$ in the deep ocean is order $200 \mathrm{~m} / \mathrm{s})$. The vertical momentum equation becomes hydrostatic, so that the density field tells us a lot about the pressure (but not everything). The resulting equations

$$
\begin{aligned}
\frac{D}{D t} u-f v & =-\frac{\partial \phi}{\partial x}+\nu \nabla^{2} u \\
\frac{D}{D t} v+f u & =-\frac{\partial \phi}{\partial y}+\nu \nabla^{2} v \\
\frac{\partial \phi}{\partial z} & =\alpha g \theta \\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} & =0 \\
\frac{D}{D t} \theta & =\mathcal{H}+\kappa \nabla^{2} \theta \\
\frac{D}{D t} & =\frac{\partial}{\partial t}+u \frac{\partial}{\partial x}+v \frac{\partial}{\partial y}+w \frac{\partial}{\partial z}
\end{aligned}
$$

This set is the most commonly used form of the equations of motion for motions on the mesoscale and up (and is not bad for submesoscale)

## Basic solutions [or problems]

We review some of the basic problems and show numerical solutions of them.

## Ekman layers

This describes the flow in the upper boundary layer under the influence of wind. The wind exerts a stress at the top

$$
\nu \frac{\partial}{\partial z} \mathbf{u}=\tau / \rho
$$

If the upper layer were just a slab it would move according to

$$
\frac{\partial}{\partial t} \mathbf{u}+\mathbf{f} \times \mathbf{u}=\frac{\boldsymbol{\tau}}{\rho H}
$$

As shown here, the slab moves to the right of the imposed force. If we think of the water as a set of slabs, as the top moves, it exerts stress on the next one which begins to move and exerts a force back on the top. As a result, the flow develops into a spiral (with superimposed inertial oscillations, as shown here. This code solves

$$
\frac{\partial}{\partial t} \mathbf{u}+\mathbf{f} \times \mathbf{u}=\frac{\partial}{\partial z} \nu \frac{\partial}{\partial z} \mathbf{u} \quad, \nu \frac{\partial}{\partial z} \mathbf{u}=\frac{\boldsymbol{\tau}}{\rho} \text { at } z=0
$$

But we can learn a lot by just looking at the vertical integral assuming the stress vanishes as $z \rightarrow-\infty$. For

$$
\mathbf{U}=\int_{-\infty}^{0} d z \mathbf{u}
$$

we have

$$
\frac{\partial}{\partial t} \mathbf{U}+\mathbf{f} \times \mathbf{U}=\frac{\boldsymbol{\tau}}{\rho}
$$

- essentially the slab equations. Thus the fluid will, on average, move 90 degrees to the right of the wind with superimposed oscillations $\cos (f t)$. I.e., the time-averaged $\mathbf{U}$ satisfies

$$
f \hat{\mathbf{z}} \times \overline{\mathbf{U}}=\frac{\boldsymbol{\tau}}{\rho} \quad \Rightarrow \quad \mathbf{U}=-\hat{\mathbf{z}} \times \frac{\boldsymbol{\tau}}{\rho f}
$$

Superimposed on this will be an oscillation like $\cos (f t)$

## Ekman pumping

If the wind stress is not spatially uniform, the Ekman transport $\mathbf{U}$ will not be constant and it can converge (driving fluid downwards) or diverge (giving upwelling).

$$
\nabla \cdot \mathbf{U}=\hat{\mathbf{z}} \cdot \nabla \times\left(\frac{\boldsymbol{\tau}}{\rho f}\right)
$$

we can connect this to the deeper flow by noting that the integrated transport in the Ekman layer satisfies

$$
\nabla \cdot \mathbf{U}+w_{e k}(0)-w_{e k}(-\infty)=0
$$

You can set $w=0$ at the top and treat the Ekman flow at the bottom as the upper boundary condition for the deep layer, or you can regard the boundary layer flow as vanishing at the bottom and require the Ekman plus interior $w$ to cancel at the top; in eaither case, we have a condition on the interior vertical velocity

$$
w=\nabla \cdot \mathbf{U}=\hat{\mathbf{z}} \cdot \nabla \times\left(\frac{\boldsymbol{\tau}}{\rho f}\right)
$$

For the simplest subtropical gyre model $\tau_{y}=0$ and $\frac{\partial}{\partial y} \tau_{x}>0$, we have $w<0$ so that fluid is pumped down into the interior.

## Mixed layer processes

Often the upper part of the ocean has nearly uniform temperature and salinity in a layer of $10-30 \mathrm{~m}$ deep in the summer reacing to 100 m or more in the winter. This is called the "mixed layer." Most of the interaction with the atmosphere and a large fraction of the photosynthesis occurs in this layer, so it is important to understand the processes occurring therein. One essential point is that "mixed" is not the same as "mixing": physical properties may remain uniform for a long time in a region which is no longer actively mixing.

## Convection

Convection occurs when less buoyant fluid overlies more buoyant fluid. Buoyancy here will be denoted $\alpha g \theta=-g \sigma$, so that the Boussinesq equations become

$$
\begin{aligned}
\frac{D}{D t} \mathbf{u}+f \hat{\mathbf{z}} \times \mathbf{u} & =-\nabla \phi+\alpha g \theta \hat{\mathbf{z}}+\nu \nabla^{2} \mathbf{u} \\
\nabla \cdot \mathbf{u} & =0 \\
\frac{D}{D t} \theta & =\kappa \nabla^{2} \theta
\end{aligned}
$$

If $\frac{\partial}{\partial z} \theta<0$, a parcel displaced upwards will be more buoyant than is surroundings and will accelerate upwards. The acceleration will be damped by friction (time order $H^{2} / \nu$ and diffusive loss of buoyancy $H^{2} / \kappa$. We expect the fluid to convect when the Rayleigh number

$$
R a=\frac{H^{4}\left|\frac{\partial}{\partial z} \alpha g \theta\right|}{\nu \kappa}
$$

is big enough (great than 100-1000 depending on boundary conditions).
As examples, we show the convection into stratification with a 2 D model; the means show the rapid erosion of the unstable stratification and the mixing into the stably stratified fluid. When cooling is applied to the surface, the mixed layer descends into the stratification with $H \sim t^{1 / 2}$ as shown by the averages - see also the waterfall view.
Kelvin-Hemlholtz
For $u=s z$ in a domain $-H / 2<z<H / 2$, the available KE is

$$
K E=\int_{-H / 2}^{H / 2} \rho_{0} u^{2} d z=\rho_{0} s^{2} \int_{-H / 2}^{H / 2} z^{2} d z=\frac{1}{12} \rho_{0} s^{2} H^{3}
$$

When the buoyancy profile is also linear $\alpha g \theta=N^{2} z$, the potential energy is lower than in the well-mixed state

$$
P E=\int_{-H / 2}^{H / 2} \rho_{0} g z-\rho_{0}\left(1-N^{2} z / g\right) z d z=\int_{-H / 2}^{H / 2} \rho_{0} g z-\rho_{0}\left(1-N^{2} z / g\right) z d z=\rho_{0} N^{2} H^{3} / 12
$$

The shear instability can overcome the stable buoyancy if $s^{2}>N^{2}$; this means the Richardson number

$$
R i=N^{2} / s^{2}
$$

has to be less than 1 . In fact, growth of perturbations requres $R i<1 / 4$. This mechanism, combined with the shears and inertial waves generated by winds allows the mixed layer to descend into a stable region even in the absence of cooling. As examples, we show $\mathbb{R i = 0 . 4}$, $R \mathrm{i}=0.2$ and $R \mathrm{i}=0$.

## Mixed layer modesls

There are a variety of mixed-layer models in use; from the biological viewpoint, they fall into two categories, ones which mix instantaneously and ones which estimate an eddy mixing coefficient. To see exactly what an eddy viscosity means, consider our equation for passive but reacting biological densities

$$
\frac{D}{D t} b_{i}=\mathcal{B}_{i}(z, t, \mathbf{b})+\nabla \cdot \kappa \nabla b_{i}
$$

under conditions where the flow is turbulent but everything is statistically horizonatlly homogemneous. The mean satisfies

$$
\frac{\partial}{\partial t} \bar{b}_{i}=\overline{\mathcal{B}_{i}\left(z, t, \overline{\mathbf{b}}+\mathbf{b}^{\prime}\right)}+\frac{\partial}{\partial z}\left(-\overline{w^{\prime} b_{i}^{\prime}}+\kappa \frac{\partial}{\partial z} \bar{b}_{i}\right)
$$

The idea behind "eddy diffusivity" is that the turbulent flux $\overline{w^{\prime} b_{i}^{\prime}}$ is linearly related to the mean gradient

$$
\overline{w^{\prime} b_{i}^{\prime}} \simeq-K(z) \frac{\partial}{\partial z} \bar{b}_{i}
$$

The equation for the fluctuations

$$
\begin{aligned}
\frac{D}{D t} b_{i}^{\prime} & =\mathcal{B}_{i}\left(z, t, \overline{\mathbf{b}}+\mathbf{b}^{\prime}\right)-\overline{\mathcal{B}_{i}\left(z, t, \overline{\mathbf{b}}+\mathbf{b}^{\prime}\right)} \\
& -w^{\prime} \frac{\partial}{\partial z} \bar{b}_{i}+\frac{\partial}{\partial z} \overline{w^{\prime} b_{i}^{\prime}}+\nabla \cdot \kappa \nabla b_{i}^{\prime}
\end{aligned}
$$

is, except for the biological dynamics term, linear in $b^{\prime}$ and forced by a term proportional to the vertical gradient of $\bar{b}_{i}$. If this gradient is uniform, and we ignore the biological terms, we can see that $b^{\prime}$ is indeed proportional to $\frac{\partial}{\partial z} \bar{b}_{i}$ so that the eddy flux must have this form. In general, however, neither the turbulence nor the gradient will be uniform, and the biological terms ma not be negligible.

Note that the biological nonlinearities also alter the means:

$$
\overline{\mathcal{B}_{i}\left(z, t, \overline{\mathbf{b}}+\mathbf{b}^{\prime}\right)} \simeq \mathcal{B}_{i}(z, t, \overline{\mathbf{b}})+\frac{1}{2} \frac{\partial^{2} \mathcal{B}_{i}}{\partial b_{j} \partial b_{k}} \overline{b_{j}^{\prime} b_{k}^{\prime}} \neq \mathcal{B}_{i}(z, t, \overline{\mathbf{b}})
$$

But let us follow the dubious assumptions through to estimate $K$. We linearize the biological term

$$
\frac{D}{D t} b_{i}^{\prime}=\frac{\partial \mathcal{B}_{i}}{\partial b_{j}} b_{j}^{\prime}-w^{\prime} \frac{\partial}{\partial z} \bar{b}_{i}+\frac{\partial}{\partial z} \overline{w^{\prime} b_{i}^{\prime}}+\nabla \cdot \kappa \nabla b_{i}^{\prime}
$$

and assume the characteristic mixing length is $h$ and scale time by $h / W^{\prime}$. Then the terms have the orders

$$
\begin{aligned}
& \frac{D}{D t} b_{i}^{\prime}=\frac{\partial \mathcal{B}_{i}}{\partial b_{j}} b_{j}^{\prime}-w^{\prime} \frac{\partial}{\partial z} \bar{b}_{i}+\frac{\partial}{\partial z} \overline{w^{\prime} b_{i}^{\prime}}+\nabla \cdot \kappa \nabla b_{i}^{\prime} \\
& W^{\prime} b^{\prime} / h \quad b^{\prime} / T_{\text {bio }} \quad W^{\prime} \bar{b} / H \quad W^{\prime} b^{\prime} / H \quad \kappa b^{\prime} / H^{2} \\
& 1 \quad h / W^{\prime} T_{\text {bio }} \quad h \bar{b} / H b^{\prime} \quad h / H \quad h^{2} / H^{2}\left(W^{\prime} h / \kappa\right)
\end{aligned}
$$

Assume the mixing length is short $(h \ll H)$ and the biological time is long compared to $h / W^{\prime}$; then $b^{\prime} \sim(h / H) \bar{b} \ll \bar{b}$, and this just reduces to the advection equation

$$
\frac{D}{D t} b_{i}^{\prime}=-w^{\prime} \frac{\partial}{\partial z} \bar{b}_{i}
$$

If we define the particle displacements as

$$
\frac{D}{D t} \zeta^{\prime}=w^{\prime}
$$

then

$$
b_{i}^{\prime}=-\zeta^{\prime} \frac{\partial}{\partial z} \bar{b}_{i}
$$

(slowly varying $\bar{b}$ and $h \ll H$ ) and the eddy flux is

$$
\overline{w^{\prime} b_{i}^{\prime}}=-\overline{w^{\prime} \zeta^{\prime}} \frac{\partial}{\partial z} \bar{b}_{i}
$$

Thus the eddy diffusivity is

$$
K=\overline{w^{\prime} \zeta^{\prime}}
$$

In the over-simplified approximation where the $\frac{\partial}{\partial t}$ term is dominant,

$$
K=\overline{w^{\prime}(t) \int^{t} d t^{\prime} w^{\prime}\left(t^{\prime}\right)}=\int^{t} \overline{w^{\prime}(t) w^{\prime}\left(t^{\prime}\right)}=\overline{w^{\prime 2}} \int_{0}^{\infty} C(\tau) d \tau
$$

with $C$ the autocorrelation function, or

$$
K=\overline{w^{\prime 2}} T_{t u r b}
$$

where we now see that the time scale is the integral time scale given by $T_{\text {turb }}=\int d \tau C(\tau)$.
More generally, we solve the trajectory equation backwards and relate $K$ to the Lagrangian autocorrelation (Taylor, 1922, but modified for inhomogeneous turbulence).
Convective adjustment:
This is the simplest variety of the instantaneous mixing models, yet a form of it is used for deep convection in polar regions. All variables are assumed to be well-mixed over a depth $H(x, y, t)$. The depth is found by the following algorithm: (1) add heating and cooling to compute the unadjusted $\alpha g \theta_{0}(z)$. If $\frac{\partial}{\partial z} \alpha g \theta<0$, we mix until there is no heavy fluid overlaying light fluid. This means that

$$
\frac{1}{H} \int_{-H}^{0} \alpha g \theta_{0}(z) d z=\alpha g \theta_{0}(-H)
$$

and

$$
\alpha g \theta= \begin{cases}\frac{1}{H} \int_{-H}^{0} \alpha g \theta_{0}(z) d z & z>-H \\ \alpha g \theta_{0}(z) & z<-H\end{cases}
$$

(2) if $\frac{\partial}{\partial z} \alpha g \theta_{0} \geq 0$, then set $H$ to some fixed or wind-stress determined depth on the order of 10 m .

PWP
The Price-Weller-Pinckel model has a similar convective adjustment step, followed by a (perhaps unnecessary) bulk Richardson number step (as in Pollard, Rhines, and Thompson), and then local mixing of neighboring layers is the Richardson number between them is less than $\frac{1}{4}$. The grid cells are partially mixed to make $R i$ supercritical; however, this will alter $\operatorname{Ri}(z \pm d z)$, so the process is iterated until all points are supercritical.

## $K($ Ri) profiles

Philander and Pacanowski developed a simple model in which the eddy diffusivity is a function of the Richardson number, becoming very large when $R i<\frac{1}{4}$ (including negative values which implies convection). The KPP model likewise produces a profile of $K$

## K-epsilon, Mellor-Yamada

The Richardson number based closures are first order in that fluxes are functionals of the mean variables. The higher order closure turbulence models, on the other hand, base their eddy viscosities on properties of the unresolved turbulence such as the eddy kinetic energy $\mathcal{K}=\frac{1}{2} \overline{\left|\mathbf{u}^{\prime}\right|^{2}}$ and the turbulent dissipation rate $\mathcal{E}=\nu \overline{\left(\nabla_{j} u_{i}^{\prime}\right)^{2}}$. From the biological point of view, the physical model is predicting the mixing rates

$$
K=\frac{\mathcal{K}^{2}}{\mathcal{E}} S_{B}\left(R i^{\prime}\right)
$$

The eddy Richardson number $R i^{\prime}=\frac{\partial \alpha g \theta}{\partial z} K^{2} / \mathcal{E}^{2}$ corrects the nondimensional function $S_{B}$ for stratification effects. Dimensional analysis dictates the form of the coefficient (and relates it to the mixing length theory of Prandtl). Mellor and Yamada predict an eddy length scale $\left(\sim \mathcal{K}^{3 / 2} / \mathcal{E}\right)$ instead. The equations for the eddy kinetic energy and dissipation rate are closed by using the eddy viscosity.

## LES

Large eddy simulations try to resolve the larger turbulent eddies and parameterize the transfers to smaller scales in a way similar to those envisioned by Kolmogorov. However, these require resolving the largest eddies on the scale of the mixed layer itself and the beginning of the 3D turbulence range.

## Internal waves and tides

When the water is stably stratified $\frac{\partial}{\partial z} \alpha g \theta>0$, a par cel move upwards will have a downward acceleration, will overshoot the initial position and end up oscillating. In the simplest version, we just have

$$
\begin{aligned}
& \frac{\partial}{\partial t} w=\alpha g \theta^{\prime} \\
& \frac{\partial}{\partial t} \alpha g \theta^{\prime}+w \frac{\partial}{\partial z} \bar{\alpha} g \theta=0 \\
& \text { or } \\
& \frac{\partial^{2}}{\partial t^{2}} w=-w \frac{\partial}{\partial z} \bar{\alpha} g \theta
\end{aligned}
$$

So that the vertical velocity oscillates

$$
w=w_{0} \cos (N t) \quad, \quad N^{2}=\frac{\partial}{\partial z} \bar{\alpha} g \theta
$$

with the Brunt-Väisälä frequency $N$. The period $2 \pi / N$ is order 10 's of minutes
For waves of the form $w=w_{0} \cos (k x+m z-\omega t)$, the dispersion relation

$$
\omega^{2}=\frac{N^{2} k^{2}+f^{2} m^{2}}{k^{2}+m^{2}}
$$

One oddity of internal waves is that the crests and troughs move in one direction (e.g., up and to the right) but the energy is transmitted in a different direction (down and to the right).

Internal waves can be generated by wind, by convection hitting the base of the mixed layer, and by flow over topography. Notably, isolated pulses can be created by tidal flow over banks.

## Eddies

As we move to larger scales, the hydrostatic approximation applies, but also the flow becomes nearly geostrophic. The terms on the horizontal momentum equation have sizes

$$
\begin{array}{ccc}
\frac{D}{D t} \mathbf{u}+f \hat{\mathbf{z}} \times \mathbf{u}= & -\nabla \phi \\
\frac{U^{2}}{L} & f U & \frac{\Phi}{L} \\
\frac{U}{f L} & 1 & 1
\end{array}
$$

with $\Phi=f U L$. For the mesoscale eddies and jets, the Rossby number $R o=U / f L$ is small and the Coriolis terms nearly balance the pressure gradients. Given the pressure, then, we can find the horizontal velocities and the buoyancy (from the hydrostatic equation).

$$
\begin{aligned}
\mathbf{u} & =\frac{1}{f} \hat{\mathbf{z}} \times \nabla \phi \\
\alpha g \theta & =\frac{\partial}{\partial z} \phi
\end{aligned}
$$

But this does not tell us how the pressure evolves - we cannot predict the changes in the flow. In addition, we'd like to know the vertical velocity since that can upwell nutrients.

To find the evolution, consider the vertical component of the vorticity (the curl of the velocity). Take the $x$ derivative of the $\frac{\partial}{\partial t} v$ equation and subtract the $y$ derivative orf the $\frac{\partial}{\partial t} u$ equation, eliminating the pressure. For large scales, however, we must account for the spherical shape of the Earth, so that $f=\hat{\mathbf{z}} \cdot 2 \boldsymbol{\Omega}=2 \omega \sin \left(\theta 0 \simeq 2 \Omega \sin \left(\theta_{0}\right)+\left[2 \Omega \cos \left(\theta_{0}\right) / a\right] y=\right.$ $f_{0}+\beta y$. Mesoscale motions have $L \ll f / \beta$. The vorticity eqn. is

$$
\frac{D}{D t}(f+\zeta)+\frac{\partial w}{\partial x} \frac{\partial v}{\partial z}-\frac{\partial w}{\partial y} \frac{\partial u}{\partial z}=(f+\zeta) \frac{\partial w}{\partial z}+\frac{\partial}{\partial z}\left(\frac{\partial \tau^{y}}{\partial x}-\frac{\partial \tau^{x}}{\partial y}\right)
$$

or

$$
\frac{D_{h}}{D t}(\zeta+\beta y)+w \frac{\partial \zeta}{\partial z}+\frac{\partial w}{\partial x} \frac{\partial v}{\partial z}-\frac{\partial w}{\partial y} \frac{\partial u}{\partial z}=(f+\zeta) \frac{\partial w}{\partial z}+\frac{\partial}{\partial z}\left(\frac{\partial \tau^{y}}{\partial x}-\frac{\partial \tau^{x}}{\partial y}\right)
$$

with $\frac{D_{h}}{D t}=\frac{\partial}{\partial t}+u \frac{\partial}{\partial x}+v \frac{\partial}{\partial y}$. For small Rossby number, $w \sim(\beta L / f)(U H / L)$ and $\zeta / f \sim$ $U / f L$; we can simplify this to

$$
\frac{D_{h}}{D t}(\zeta+\beta y)=f \frac{\partial w}{\partial z}+\frac{\partial}{\partial z}\left(\frac{\partial \tau^{y}}{\partial x}-\frac{\partial \tau^{x}}{\partial y}\right)
$$

We split the stratification into a mean part and horizontally/ temporally varying parts $\alpha g \theta=\int^{z} N^{2}+\alpha g \theta^{\prime}$. From geostropic and hydrostatic balance, $\alpha g \theta^{\prime} \sim f U L / H$ and is order $(U / f L)\left(f^{2} L^{2} / N^{2} H^{2}\right)$ compared to the vertical variations of the mean stratification. For the mesoscale, the last term is order one, and the buoyancy equation becomes

$$
\frac{D_{h}}{D t} \alpha g \theta^{\prime}+w N^{2}=Q
$$

We can eliminate $w$ to find an approximate conserved quantity

$$
\frac{D_{h}}{D t}\left(\zeta+\beta y+\frac{f}{N^{2}} \alpha g \theta^{\prime}\right)=\frac{\partial}{\partial z} \operatorname{curl} \boldsymbol{\tau}+\frac{\partial}{\partial z} \frac{f Q}{N^{2}}
$$

the quasi-geostrophic potential vorticity. The velocities and vorticity are computed from the pressure using the geostrphic relation; therefore this equation tells us how the pressure will change.

If, insteady, we use the thermal wind relationship in the form

$$
f \frac{\partial}{\partial z} \zeta=\nabla^{2} \alpha g \theta^{\prime}
$$

we can eliminate the $\frac{\partial}{\partial t}$ term and find an equation telling us what the vertical velocity is

$$
N^{2} \nabla^{2} w+f^{2} \frac{\partial^{2}}{\partial z^{2}} w=-\nabla^{2}\left(\mathbf{u} \cdot \nabla \alpha g \theta^{\prime}\right)+f \frac{\partial}{\partial z}[\mathbf{u} \cdot \nabla(\zeta+\beta y)]+\nabla^{2} Q-f \frac{\partial^{2}}{\partial z^{2}} \operatorname{curl} \boldsymbol{\tau}
$$

with the first two terms determined by the pressure field. Some examples of eddy movement are shown; nearly linear with velocity independent of depth or surface intensified, and nonlinear cases.

## Sverdrup flow/thermocline

The standard approach assumes the interior motions have the same very large scale as the wind forcing. Then geostrophy folds, but the assumption $\beta L / f \ll 1$ and $N^{2} H^{2} / f^{2} L^{2} \sim$ 1 no longer hold. We get information about the vertical velocity by substituting the geostrophic relations

$$
\mathbf{u}=\frac{1}{f} \hat{\mathbf{z}} \times \nabla \phi
$$

into the continuity equation. Then the divergence of the horizontal flow is

$$
\nabla \cdot \frac{1}{f} \hat{\mathbf{z}} \times \nabla \phi=-\frac{\beta}{f} v=-\frac{\partial w}{\partial z}
$$

and we get the Sverdrup relation

$$
\beta v=f \frac{\partial w}{\partial z}
$$

which can be integrated using the Ekman pumping relation and $w=0$ at the bottom (or some depth)

$$
\beta \int_{-H}^{0} d z v=f \hat{\mathbf{z}} \cdot \nabla \times\left(\frac{\boldsymbol{\tau}}{\rho f}\right)
$$

The simulation here shows that the Sverdrup flow first appears throughout the depth; however, waves from the eastrn boundary shut off the deeper flow and concentrate the transport into the upper layer. We can examine the upper and lower zonal flow or the $\mathrm{V}(\mathrm{x}, \mathrm{t})$ figures to see this. $\boldsymbol{\theta}$

## Western boundary currents

The experiment above shows that the wind-induced flow is southward in the subtropical gyre; it must go back northward somehwere. The signals propagate to the west, suggesting that it should occur in a western boundary current which has a scale $\ell$ such that the flow is not completely described by geostrophy.

The wind puts in clockwise swirl; somewhere we must have a source of counterclockwise swirl; that can occur via friction on the western boundary but not the eastern wall. In the vorticity equation the wind stress (in the subtropical gyre where $\frac{\partial}{\partial y} \tau^{x}>0$ ) is putting in negative vorticity; somewhere er need a source of positive (counter-clockwise spin) $\frac{D_{h}}{D t} \zeta$ to balance it off. If we integrate over depth and over the basin, the only term left is the bottom friction. For a nortward flowing current, the bottom drag will be exerting a southward force on the fluid; for a WBC, this corresponds to counter-clockwise while for an EBC it would increse the clockwise forcing. Similar arguments apply for side-wall friction and in the subpolar gyre.

