Eddy Dynamics

Transport

Eddies effect[†]

- horizontal transport
- vertical transport

of the biota. We want to understand both local and averaged effects of these processes.

Vertical transport

Much of the interest in vertical transport focuses on the surface layer, though it can also be important in establishing deep structure.

A simplified model

We are going to build a simplified model of the vertical density structure, consisting of a mixed layer, a homogeneous layer representing the water above the thermocline, and a second denser layer below the thermocline. sketch

The pressures satisfy

$$\begin{aligned} p_m &= \rho_m g(h_1 + h_2 - z) \\ p_1 &= \rho_m g h_m + \rho_1 g(h_1 + h_2 - h_m - z) \\ p_2 &= \rho_m g h_m + \rho_1 g(h_1 - h_m) + \rho_2 g(h_2 - z) \end{aligned}$$

and, under the assumption that the mixed layer depth is fixed, the gradients of the dynamic pressure $\phi = p/\rho$ become

$$\begin{aligned} \nabla\phi_m &= g\nabla(h_1 + h_2) \\ \nabla\phi_1 &= g\nabla(h_1 + h_2) \\ \nabla\phi_2 &= \frac{\rho_1}{\rho_2} g\nabla h_1 + g\nabla h_2 \end{aligned}$$

or

$$\nabla\phi_1 = \nabla\phi_m = \nabla\phi_2 + g\frac{\rho_2 - \rho_1}{\rho_2}\nabla h_1$$

Momentum equations:

$$\frac{\partial}{\partial t}\mathbf{u}_i + \mathbf{u}_i \cdot \nabla\mathbf{u}_i + f\hat{\mathbf{z}} \times \mathbf{u}_i = -\nabla\phi_i$$

The pressure gradients are independent of z within a layer, so the velocities likewise can depend only on x and y.

Mass conservation:

[†] If you ever have me on a committee, be sure to use effect and affect properly! The usage here is unusual but correct: effect(v)= to bring about. Avoid it and stick to the effect(n) or affect(v) forms.

Examining the integral of the mass equation across the bottom layer gives

$$\frac{\partial}{\partial t}\int_0^{h_2}\rho_2 + \nabla\cdot\int_0^{h_2}\mathbf{u}_2\rho_2 = \rho_2\frac{D}{Dt}h_2 - w\rho_2\Big|_0^{h_2}$$

or

$$\frac{\partial}{\partial t}h_2 + \nabla\cdot h_2\mathbf{u}_2 = \frac{D}{Dt}h_2 - w(h_2)$$

Taking the thermocline to be a material surface gives $w_2 = \frac{D}{Dt}h_2$ and

$$\frac{\partial}{\partial t}h_2 + \nabla\cdot h_2\mathbf{u}_2 = 0$$

Similar equations integrating across a uniform water mass from h_a to h_b give

$$\frac{\partial}{\partial t}(h_b - h_a) + \nabla\cdot(h_b - h_a)\mathbf{u} = \frac{D}{Dt}(h_b - h_a) + w(h_a) - w(h_b)$$

Applying for $h_a = h_2$, $h_b = h_2 + h_1 - h_m$ and for $h_a = h_2 + h_1 - h_m$, $h_b = h_a + h_2$, summing (using $\mathbf{u} = \mathbf{u}_1$ in both the ML and upper layer) and inserting $w(h_2) = \frac{D}{Dt}h_2$, $w(h_1 + h_2) = \frac{D}{Dt}(h_1 + h_2)$ gives

$$\frac{\partial}{\partial t}h_1 + \nabla\cdot h_1\mathbf{u}_1 = 0$$

However, looking just at the ML gives

$$\begin{aligned} h_m\nabla\cdot u_1 &= w(h_2 + h_1 - h_m) - w(h_2 + h_1) \\ &= w(h_2 + h_1 - h_m) - \frac{D}{Dt}(h_2 + h_1 - h_m) \\ &\equiv w_e \end{aligned}$$

with w_e the entrainment velocity across the MLB. The stretching associated with the entrainment is therefore

$$\frac{w_e}{h_m} = \nabla\cdot\mathbf{u}_1 = -\frac{1}{h_1}\frac{D}{Dt}h_1$$

When the thermocline is shallowing, water is being pushed into the ML (or up and the ML is mixing back down) so that $w_e > 0$.

Mixed-layer biology

We start with

$$\frac{\partial}{\partial t} b_i + \nabla \cdot \mathbf{u} b_i + \frac{\partial}{\partial z} \tilde{w} b_i = \mathcal{B}_i + \frac{\partial}{\partial z} K \frac{\partial}{\partial z} b_i$$

in the ML. Here the vertical velocity includes swimming or sinking. Integrating over the mixed layer gives

$$\frac{\partial}{\partial t} \int_{h_0}^{h_0+h_m} b_i + \nabla \cdot \int_{h_0}^{h_0+h_m} \mathbf{u}_1 b_i =$$

$$b_i(h_0+h_m) \left[\frac{D}{Dt}(h_0+h_m) - \tilde{w}(h_0+h_m) \right] - b_i(h_0) \left[\frac{D}{Dt} h_0 - \tilde{w}(h_0) \right]$$

$$+ \int_{h_0}^{h_0+h_m} \mathcal{B}_i + K \frac{\partial}{\partial z} b_i \Big|_{h_0+h_m} - K \frac{\partial}{\partial z} b_i \Big|_{h_0}$$

Taking b_i to be well-mixed within the layer and using the impermeability of the sea-surface gives

$$\frac{\partial}{\partial t} h_m b_i + \nabla \cdot h_m \mathbf{u}_1 b_i = b_i^*(h_0) \tilde{w}_e + \int_{h_0}^{h_0+h_m} \mathcal{B}_i + \lambda \left[b_i(h_0^-) - b_i \right]$$

The notation b_i^* is used as an "upwind" difference: if $\tilde{w}_e > 0$ then $b_i^* = b(h_0^-)$, the value in the layer just beneath the interface. If $\tilde{w}_e < 0$ then $b_i^* = b_i$. The term with λ represents the small-scale turbulent diffusive flux $K/H_{transition}$. In this form, the entrainment and biological movement such as sinking do not change the integrated values.

The equation above is useful for examining the net biomass, assuming it is almost all in the mixed layer. We can also write an eqn. for the concentration

$$\frac{\partial}{\partial t} b_i + \mathbf{u}_1 \cdot \nabla b_i = b_i^*(h_0) \frac{\tilde{w}_e}{h_m} - b_i \frac{w_e}{h_m} + \frac{1}{h_m} \int_{h_0}^{h_0+h_m} \mathcal{B}_i + \frac{\lambda}{h_m} \left[b_i(h_0^-) - b_i \right]$$

For variables confined to the mixed layer with no independent motion, the flux across the sea surface will be zero and $\tilde{w}_e = w_e$, giving

$$\frac{\partial}{\partial t} b_i + \mathbf{u}_1 \cdot \nabla b_i = \frac{w_e}{h_m} \left[b_i^*(h_0) - b_i \right] + \frac{1}{h_m} \int_{h_0}^{h_0+h_m} \mathcal{B}_i - \frac{\lambda}{h_m} b_i$$

with $b_i^*(h_0) = 0$ for $w_e > 0$ (dilution) and $b_i^*(h_0) = b_i$ for $w_e < 0$ (advection in and down where it disappears because of the biological terms). Here $h_0 = h_1 + h_2 - h_m$. For the deep values to be zero, the mixed layer should be intermediate in depth

$$\frac{1}{h_m} \int_{h_0}^{h_0+h_m} dz\, \mu(z) > d_p > \mu(h_0)$$

The stretching is just $w_e/h_m = -\frac{D}{Dt} \ln h_1$.

We can estimate w/h from the vertical vorticity equation:

$$\frac{D}{Dt}(\zeta + f) = (\nabla \times \mathbf{u} + f\hat{\mathbf{z}}) \cdot \nabla w$$

giving

$$\frac{\partial w}{\partial z} \sim \frac{1}{f}\frac{D}{Dt}\zeta \sim \frac{U^2}{fL^2}$$

For $U = 10\ km/d$, $L = 5\ km$ (frontal scale), $f = 8/d$, this is $0.5/d$ which is comparable to the fast biological time scales. But it is an overestimate, in that the advection depends on the along-front scales which will be larger (but the cross front scales may be smaller). Mahadevan *et al.*(2008) estimate vertical velocities on the edges of eddy fronts to be as large as 100 m/d giving even larger peak values of $\frac{\partial w}{\partial z}$.

Eddy transport and mixing

Eddies are thought to provide an "eddy diffusivity" mixing up the larger-scale means. Let us examine the argument in some detail. Suppose we split the fields up into means and eddy (or eddy-induced) parts

$$\frac{\partial}{\partial t}\overline{b}_i + \overline{\mathbf{u}} \cdot \nabla \overline{b}_i + \nabla \cdot \overline{\mathbf{u}' b_i'} = \overline{\mathcal{B}_i(\overline{b} + b')}$$

Stirring and averaging
Stirring
Mixing
Both

Potential vorticity

We find the vorticity equation bu cross-differentiating the momentum equations

$$\frac{D}{Dt}u - fv = -\frac{\partial}{\partial x}\phi$$
$$\frac{D}{Dt}v + fu = -\frac{\partial}{\partial y}\phi$$

to eliminate the pressure. The result is

$$\frac{D}{Dt}(\zeta + f) + (\zeta + f)\nabla \cdot \mathbf{u} = 0$$

Combining with the mass equation gives

$$\frac{D}{Dt}(\zeta + f) - q\frac{D}{Dt}h = 0 \quad or \quad \frac{D}{Dt}qh - q\frac{D}{Dt}h = 0 \quad \Rightarrow \quad \frac{D}{Dt}q = 0$$

with $q = (\zeta + f)/h$ being the "potential vorticity" – a conserved scalar property of the flow in each layer.

Nearly geostrophic motions

Rossby waves

The hydrostatic equations are

$$\frac{D}{Dt}u_j - fv_j = -\frac{\partial}{\partial x}\phi_j$$
$$\frac{D}{Dt}v_j + fu_j = -\frac{\partial}{\partial y}\phi_j$$
$$\frac{D}{Dt}h_j + h_j\nabla\cdot\mathbf{u}_j = 0$$
$$\phi_1 = \phi_2 + g'h_1$$

with $g' = g(\rho_2 - \rho_1)/\rho_2 << g$ and $f = f_0 + \beta y$.

In the simplest case ("barotropic"), we can find solutions

$$v_j = V\cos k(x - ct) \quad , \quad u_j = \delta h_j = 0$$

The pressure balances the coriolis force

$$\phi_j = f\frac{V}{k}\sin k(x - ct)$$

Note that the pressure has a y variation because of β: the pressure signals are stronger in the north. This leads to an acceleration

$$\frac{\partial}{\partial t}v\ [= kcV\sin k(x - ct)] = -\frac{\partial}{\partial y}\phi = -\beta\frac{V}{k}\sin k(x - ct)$$

giving

$$c = -\frac{\beta}{k^2}$$

We find westward propagating waves with the speed decreasing for shorter waves.

The baroclinic case is similar but is only approximate having very small $u, \delta h$. However it is more interesting because it has non-zero vertical velocities (or stretching $s = -w_e/h_m = \frac{D}{Dt}\ln h_1 \sim \frac{1}{H_1}\frac{\partial}{\partial t}h_1$). We can simplify the potential vorticity by setting

$$Q = Hq - f_0 = \frac{H}{h}\left[\zeta + \beta y - f_0\frac{h - H}{H}\right]$$

and then approximate it by dropping the factor H/h. The PV conservation equation in the linearized form becomes

$$\frac{\partial}{\partial t}\left[\zeta' - f_0\frac{h'}{H}\right] + \beta v' = 0$$

For $v_1' = V\cos k(x-ct)$,

$$\frac{\partial}{\partial t}\zeta_1' = -c\frac{\partial}{\partial x}\zeta_1' = -c\frac{\partial^2}{\partial x^2}v_1' = ck^2 v_1'$$

and

$$-\frac{\partial}{\partial t}f_0\frac{h_1'}{H_1} = c\frac{f_0}{H_1}\frac{\partial}{\partial x}h_1'$$

The hydrostatic equations imply

$$g'\frac{\partial}{\partial x}h' = \frac{\partial}{\partial x}(\phi_1' - \phi_2') = f_0(v_1' - v_2')$$

We look for solutions with no net transport $v_1 H_1 = -v_2 H_2$ so that

$$\frac{\partial}{\partial x}h' = \frac{f_0(H_1+H_2)}{g'H_2}v_1$$

and the PV equation becomes

$$ck^2 v_1 + c\frac{f_0^2(H_1+H_2)}{g'H_1H_2}v_1 + \beta v_1 = 0$$

The same equation holds from the lower layer equations as long as the stretching from the SSH is neglected $h_2' \simeq -h_1'$.

From this equation we get the propagation rate

$$c = -\frac{\beta}{k^2 + 1/R_d^2}$$

with the "deformation radius"

$$R_d^2 = \frac{g'H_1H_2}{f^2(H_1+H_2)}$$

These waves also propagate westward, but more slowly and with a maximum speed of $-\beta R_d^2$. In the ocean $R_d \sim 40\,km$ and $\beta \sim 2\times 10^{-11}\,m^{-1}s^{-1} = 1.7\times 10^{-3}\,km^{-1}d^{-1}$ giving a speed order $3\,km/d$.

Winds excite both kinds of waves, but at scales with $k \sim 1/1000\,km$ giving barotropic speeds order $2000\,km/d$. These can cross a $20,000\,km$ basin in $10\,d$ whereas a baroclinic wave can take several decades.

However, the baroclinic motions are those which have substantial thermocline displacements and vertical velocities. For $H_1 = 0.8\,km$, $H_2 = 4\,km$, $g' \sim 1.3\times 10^5\,km/d^2 \sim 0.02 m/s^2$. For $v_1 = 20\,km/d$ and a $40\,km$ length scale, $\phi_1 \sim 6400\,km/d$ and $\delta h_1 \sim 50\,m$. This gives a stretching $w_e/h_m \sim c\frac{\partial}{\partial x}(\delta h_1/H_1) \sim 1/200\,d$. This is really an underestimate, since the changes associated with advection are much stronger. As mentioned above (but with a shorter length scale) the vorticity equation suggests $U^2/L^2 - \sim f\frac{D}{Dt}\ln h$ giving a stretching time scale order $30\,d$ here. For Rings, $v_1 \sim 100\,km/d$ so even the propagation estimate gives a $40\,d$ time scale. Example Example with 1/40

Isolated eddies

Isolated eddies such as Rings are characterized by the flow speeds being large compared to the propagation speeds. As a simple model, ignore the deep flow so that $g'h_1 = g'H_1 + \phi_1$ and

$$\frac{D}{Dt}\mathbf{u}_1 + f\hat{\mathbf{z}} \times \mathbf{u}_1 = -\nabla\phi_1$$

$$\frac{\partial}{\partial t}\phi_1 + \nabla \cdot (g'H_1 + \phi_1)\mathbf{u}_1 = 0$$

To a reasonable approximation, $\mathbf{u}_1 = \frac{1}{f_0}\hat{\mathbf{z}} \times \phi_1$ and the PV equation (replacing H/h with 1)

$$\frac{D}{Dt}Q = 0 \quad , \quad Q = \zeta_1 + \beta y - \frac{f_0\phi_1}{g'H_1}$$

can be written as

$$\frac{\partial}{\partial t}Q + \frac{\partial\psi}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial Q}{\partial x} = 0 \quad , \quad \psi = \frac{1}{f_0}\phi_1 \quad , \quad Q = \nabla^2\psi - \frac{1}{R_d^2}\psi + \beta y$$

Movement

$$\frac{\partial}{\partial t}Q' + \frac{\partial\psi}{\partial x}\frac{\partial Q'}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial Q'}{\partial x} + \beta\frac{\partial\psi}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\int d\mathbf{x}\, Q' = 0 \quad , \quad \frac{\partial}{\partial t}\int d\mathbf{x}\, xQ' = \beta\int d\mathbf{x}, \psi$$

But

$$\int d\mathbf{x}\, Q' = -\frac{1}{R_d^2}\int d\mathbf{x}\, \psi$$

so

$$\frac{\partial}{\partial t}\int d\mathbf{x}\, xQ = -\beta R_d^2\int d\mathbf{x}\, Q$$

and the center of PV moves westward at the long wave speed. However, a wave pattern is also generated so that the peak value of Q' is not necessarily at the center of mass.

Transport

If we approximate it as moving steadily, then there is a stagnation point where $\mathbf{u} = \mathbf{c}$; in the moving frame, the streamlines form a loop. Within that loop, fluid is transported with the eddy. Variability causes leakage out and slow loss of properties.

Upwelling

With this form, the stretching simplifies since $h_1 \propto \psi$

$$s = -\frac{1}{H_1}\frac{D}{Dt}h_1 = -\frac{1}{H_1}\frac{\partial}{\partial t}h_1 \simeq c\frac{\partial}{\partial x}h_1$$

But we have to remember that the nutrients, for example, satisfy

$$\frac{D}{Dt}N = -uptake + remin + \frac{w^+}{h_m}[N_0 - N]$$

gives a net input in the upwelling region integrated along the track (sketch. Dynamically, the fields are more complex: phys bio

Generation – baroclinic instability

Consider the following model of a jet:

$$\overline{q}_1 = \frac{1}{2}\left[\frac{f}{h_s} + \frac{f}{h_n}\right] + \frac{1}{2}\left[\frac{f}{h_n} - \frac{f}{h_s}\right]\tanh(y/W)$$

This has a geostrophically balanced flow

$$\overline{q} = \frac{f - \frac{\partial}{\partial y}\overline{u}}{\overline{h}} = \frac{f + (g'/f)\frac{\partial^2}{\partial y^2}\overline{h}}{\overline{h}}$$

or

$$\frac{\partial^2}{\partial y^2}\overline{h} - \frac{f\overline{q}}{g'}\overline{h} = -f$$

Perturbations satisfy

$$\frac{\partial}{\partial t}q' + v'\frac{\partial}{\partial y}\overline{q} = 0 \quad , \quad \overline{h}q' = \zeta' - \overline{q}h'$$

Note that $\overline{q}_2 = f/(H - \overline{h})$ (no deep jet).

For

$$u' = -\frac{\partial}{\partial y}\psi - \frac{\partial}{\partial x}\Phi \quad , \quad v' = \frac{\partial}{\partial x}\psi - \frac{\partial}{\partial y}\Phi$$

and using the simplest geostrophic balance for h (taking the divergent flow to be small comparted to the rotational flow) gives

$$\overline{h}q_1' = \nabla^2\psi_1 - \frac{f\overline{q}_1}{g'}(\psi_1 - \psi_2)$$

and $\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}$; the result is coupled equations in ψ_1, ψ_2.

Growth rate

Stretching

Other forcings include winds, baroclinic and barotropic instability, topography

SQG

Submesoscale